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# On the Relationship Between the Vogler Algorithm Derivation and the Parabolic Equation for Multiple Knife-Edge Diffraction

Viet-Dung Nguyen, Huy Phan, Oliver Y. Chén, Ali Mansour, Arnaud Coatanhay, and Thierry Marsault

**Abstract**—We prove the equivalence between the derivation of the Vogler algorithm and the parabolic equation for multiple knife-edge diffraction problems under certain conditions. To illustrate the key ideas, we present two case studies for single and double knife-edge diffraction, where the proposed results generalize the available results in the literature. The insight from our proof can serve two purposes: allowing cross-checking results of the Vogler algorithm and the parabolic equation; and building the foundation on which new methods for solving this essential problem and related ones can be developed.

**Index Terms**—Multiple knife-edge diffraction, parabolic equation, RF transmission, Vogler method.

## I. INTRODUCTION

Multiple knife-edge diffraction is a fundamental problem for analysis and designing reliable systems in radar and wireless communication applications [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], especially in the presence of irregular terrain. Generally, this problem can be formulated and solved in two different ways by using the Vogler algorithm [1], [2], [3] or solving the parabolic equation (PE) [4], [8], [9].

The parabolic equation is used as an influential tool to model electromagnetic propagation, in general, and diffraction, in particular [4], [8], [9], [12]. The Vogler algorithm, meanwhile, has proved its outstanding accuracy compared to existing solutions in the literature [13].

From theoretical aspects, a link between the parabolic equation and the Fresnel diffraction formula for double knife-edge diffraction is first established in [4] (see Chapters 2 and 7 as well as references therein). Shanin and Korolkov [14] showed that the solution of the parabolic equation for half-plane diffraction is a good estimation of the exact solution of the Helmholtz equation, thus explaining why the solution of the parabolic equation works well even close to the edge of the half-plane. In [15], a system of integral equations in a spectrum domain is derived from the parabolic equation to tackle multiple knife-edge diffraction.

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From practical aspects, while the validation of the parabolic equation method with experiment data is well observed as presented in [4], [16], [17], and [18] and references therein, there is a lack of numerical comparison of two approaches as well as its agreement is not fully understood.

To fill this gap, we present a proof to show that those approaches are two different solutions of the same problem. This proof is built on the results of [4] for the double knife-edge model. Specifically, we generalize the previous results to an arbitrary number of knife edges,  $N \geq 1$ , where we present the main result in Section II and two case studies with  $N = 1$  and 2 in Section III. The numerical comparison and validation are presented in Supplementary Material with this communication. We note that our actual study focuses on the analytical derivation aspect from the point of view of the parabolic equation which is significantly different from [3], [19] that aim to solve the diffraction integrals efficiently in different setups.

Our proof could bring insights in terms of both theoretical and practical aspects. We can cross-check the obtained results with two different methods or provide alternating methods. We could observe that the result from the Vogler method can play the role of an independent double-check for that of a parabolic equation, and vice versa.

Aiming to solve the same equation, the Vogler method provides a numerical solution based on a complicated “integral form,” while the main approaches to solving parabolic equations are from a “differential equation form.” Following [4] and recent surveys [8], [9], two main approaches for solving parabolic equations numerically are the split-step-Fourier and finite-difference methods that are expected to have different characteristics compared to the Vogler algorithm. In the literature, there are certain statements that connect the parabolic equation and the Vogler algorithm, for example, see the second paragraph of Section II in [20]. The result, however, does not apply to a general case (i.e., implicitly for triple knife-edge diffraction), and, to the best of our knowledge, there is no explicit theory for that case. Moreover, the proposed theoretical results here can be used as additional conditions to develop new methods, for example, solving multiple knife-edge diffraction by physics-aided machine learning [21] which has recently attracted significant attention.

## II. VOGLER ALGORITHM DERIVATION AND PARABOLIC EQUATION WITH MULTIPLE KNIFE-EDGE DIFFRACTION

We first give a brief background on the parabolic equation and then outline the main steps of the proof. Our main contributions are related to a generalization of the result presented in [4], from  $N = 1, 2$  to an arbitrary  $N \geq 1$ . We consider the scalar wave [4]

$$\frac{\partial^2 u(x, z)}{\partial x^2} + \frac{\partial^2 u(x, z)}{\partial z^2} + 2ik \frac{\partial u(x, z)}{\partial x} + (n^2(x, z) - 1)k^2 u(x, z) = 0 \quad (1)$$

where the  $x$ -axis defines the range or the paraxial direction and the  $z$ -axis refers to height (see Fig. 1). The refractive index  $n$  is a function

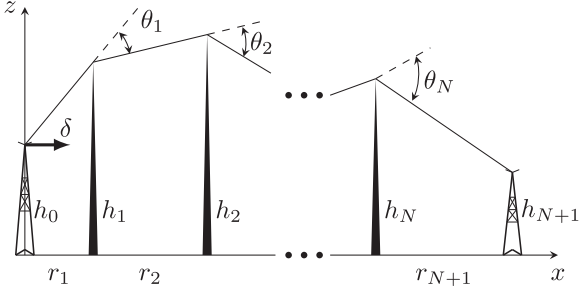


Fig. 1. Geometry of multiple knife edges with a point (or punctual) source  $\delta$  at a transmitter.

of  $x$  and  $z$ . The wave and complex numbers are defined as  $k = 2\pi/\lambda$  and  $i = (-1)^{1/2}$ , respectively. The standard parabolic equation is then given by

$$\frac{\partial^2 u(x, z)}{\partial z^2} + 2ik \frac{\partial u(x, z)}{\partial x} + k^2(n^2(x, z) - 1)u(x, z) = 0. \quad (2)$$

We note that (2) is achieved by using the first-order approximation of the forward wave after factorizing (1). If we assume that the propagation medium is in a vacuum or a relatively homogeneous one, that is,  $n(x, z) \approx 1$ , we then can simplify (2) as follows:

$$\frac{\partial^2 u(x, z)}{\partial z^2} + 2ik \frac{\partial u(x, z)}{\partial x} = 0. \quad (3)$$

Now, we present the main steps of the proof.

- 1) The closed-form solution of the parabolic equation in vacuum or a relatively homogeneous medium (see Lemma 1).
- 2) Derivation of the reduced field multiple knife-edge diffraction from the parabolic equation (see Theorem 1).
- 3) Derivation of the multiple knife-edge diffraction from the Fresnel–Kirchhoff theorem and the equivalence between the two derivations (see Theorem 2).

*Lemma 1:* In the case of propagation in vacuum or in a relatively homogeneous medium where  $x > 0$  and  $n(x, z) \approx 1$ , a closed-form solution of (3) is given by

$$u(x, z) = \sqrt{\frac{1}{\lambda x}} \exp\left(-\frac{i\pi}{4}\right) \times \int_{-\infty}^{+\infty} u(0, v) \exp\left(\frac{ik(z-v)^2}{2x}\right) dv. \quad (4)$$

*Proof:* See Appendix A.  $\square$

From Lemma 1, we can observe that the solution of a given range can be deduced from the initial field  $u(0, v)$ . This solution is computed by, first, transforming into the frequency domain and then returning to the original Cartesian domain. Here, using the Fourier transform (see Appendix A) simplifies the computation.

*Lemma 2:* Considering a small diffraction angle approximation, the total path length difference of  $N$  knife-edges,  $d_N$ , can be approximated as follows:

$$d_N \approx \sum_{m=0}^N \frac{(z_{m+1} - z_m)^2}{2r_{m+1}} \quad (5)$$

where, by convention,  $z_0 = h_0$  and  $z_{N+1} = h_{N+1}$ .

*Proof:* See Appendix B.  $\square$

It is worth mentioning that Lemma 2 serves not only as an immediate step, but also a specific condition (i.e., small-angle diffraction approximation) to obtain Theorem 1.

*Theorem 1:* Given the multiple knife-edge geometry and using small-angle diffraction approximation, the reduced field  $u(x, z)$  at the receiver is given by

$$u\left(\sum_{m=1}^{N+1} r_m, h_{N+1}\right) = \text{const}_N \int_{h_1}^{+\infty} \cdots \int_{h_N}^{+\infty} \exp(ikd_N) dz_1 \dots dz_N \quad (6)$$

where  $\text{const}_N$  is a constant.

*Proof:* See Appendix C.  $\square$

*Theorem 2:* Given the multiple knife-edge geometry and the total path length difference  $d_N$  in (5), solving a solution of the parabolic equation in vacuum or a relatively homogeneous medium is equivalent to the computation of the Fresnel–Kirchhoff formulation, that is, the diffraction attenuation  $A_N$  between the total field  $E$  and the free-space field  $E_0$

$$A_N = \frac{E}{E_0} = \frac{E(h_1, \dots, h_N)}{E(-\infty, \dots, -\infty)} \quad (7)$$

where

$$E(h_1, \dots, h_N) = \int_{h_1}^{+\infty} \cdots \int_{h_N}^{+\infty} \exp(ikd_N) dz_1 \dots dz_N. \quad (8)$$

*Proof:* By applying Theorem 1 and comparing (8) and (6),  $E$  is proportional to  $u$ , that is,  $E \propto u$ . Moreover, note that the free-space field can be computed by setting the lower limits of (6) to  $-\infty$  (i.e.,  $h_1 = \dots = h_N \rightarrow -\infty$ ), we achieve (7). After that, let us consider multiple knife-edge diffraction geometry presented in Fig. 1. The Fresnel–Kirchhoff formulation of the field  $E$  to the free-space  $E_0$  is given in [2]

$$A_N = \frac{E}{E_0} = \frac{\int_{h_1}^{+\infty} \cdots \int_{h_N}^{+\infty} \exp(ikd_N) dz_1 \dots dz_N}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp(ikd_N) dz_1 \dots dz_N} \quad (9)$$

where the Huygens' principle is applied [22], [23] (i.e., we integrate over each aperture, from  $h_m$  to  $\infty$ ). We note that edge currents and backscatter are disregarded in this derivation. We thus conclude the proof.  $\square$

In [1] and [2], Vogler proposed an efficient method, so-called the recursive Vogler algorithm, to compute the integral (7) directly. Hereinafter, we only summarize the main steps. The essential idea is to transform the  $N$ -fold integral into  $N$  integrals. To obtain a closed-form solution, we need to eliminate the cross-product terms between  $z_m$  and  $z_{m+1}$  in (5) (or rewrite (5) as the sum of squares) by changing variables so that we achieve a product of  $N$  integrals. This is possible because the lower limits and the upper limits of (8) are infinity (see Lemma 3 for a general case). For more details, see [2] and [3].

*Lemma 3:* The free-space field  $E_0$  is given by

$$E_0 \propto \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp(ikd_N) dz_1 \dots dz_N \\ = (\sqrt{\pi})^N \exp\left(ik \frac{(h_0 - h_{N+1})^2}{2 \sum_{m=1}^{N+1} r_m}\right) \quad (10)$$

where  $A \propto B$  means that  $A$  is proportional to  $B$ .

### III. CASE STUDIES

To illustrate the main steps, we present here two case studies for  $N = 1$  and 2. For each case, we found that the result is consistent and generalizes the existing results in the literature.

### A. Case Study 1: Single Knife-Edge Diffraction

When  $N = 1$ , we apply Lemma 1 (see also the first part of proof of Theorem 1 in Appendix C) and obtain

$$u(r_1 + r_2, h_2) = \frac{1}{\lambda \sqrt{r_1(r_1 + r_2)}} \exp\left(-\frac{i\pi}{2}\right) \times \int_{h_1}^{+\infty} \exp\left(\frac{ik}{2} \left( \frac{(z_1 - h_0)^2}{r_1} + \frac{(h_2 - z_1)^2}{r_2} \right)\right) dz_1. \quad (11)$$

Thus, the diffraction field,  $A_1$ , in this case is given by

$$A_1 = \frac{E}{E_0} = \frac{\int_{h_1}^{+\infty} \exp\left(i\frac{\pi}{\lambda} \left( \frac{(z_1 - h_0)^2}{r_1} + \frac{(h_2 - z_1)^2}{r_2} \right)\right) dz_1}{\int_{-\infty}^{+\infty} \exp\left(i\frac{\pi}{\lambda} \left( \frac{(z_1 - h_0)^2}{r_1} + \frac{(h_2 - z_1)^2}{r_2} \right)\right) dz_1}. \quad (12)$$

We now consider the integral in the numerator of  $A_1$ . Notice that

$$\frac{(z_1 - h_0)^2}{r_1} + \frac{(h_2 - z_1)^2}{r_2} = \left[ \sqrt{\frac{1}{r_1} + \frac{1}{r_2}} z_1 - \frac{\left(\frac{h_0}{r_1} + \frac{h_2}{r_2}\right)}{\sqrt{\frac{1}{r_1} + \frac{1}{r_2}}} \right]^2 + \frac{(h_2 - h_0)^2}{r_1 + r_2}. \quad (13)$$

Let

$$\theta_1 = \frac{(h_1 - h_0)}{r_1} + \frac{h_1 - h_2}{r_2}, \quad v_1 = \theta_1 \sqrt{\frac{2}{\lambda} \frac{r_1 r_2}{r_1 + r_2}}.$$

By changing the variable

$$w = \sqrt{\frac{2}{\lambda}} \left( \sqrt{\frac{1}{r_1} + \frac{1}{r_2}} z_1 - \frac{\left(\frac{h_0}{r_1} + \frac{h_2}{r_2}\right)}{\sqrt{\frac{1}{r_1} + \frac{1}{r_2}}} \right)$$

we obtain

$$dw = \sqrt{\frac{2}{\lambda}} \sqrt{\frac{1}{r_1} + \frac{1}{r_2}} dz_1$$

and  $w \rightarrow v_1$  when  $z_1 \rightarrow h_1$  as well as  $u \rightarrow \infty$  when  $z_1 \rightarrow \infty$ . Thus, we can write the total field  $E$  as follows:

$$E \propto \frac{1}{\sqrt{\frac{2}{\lambda}} \sqrt{\frac{1}{r_1} + \frac{1}{r_2}}} \exp\left(i\frac{\pi}{\lambda} \frac{(h_2 - h_0)^2}{r_1 + r_2}\right) \times \int_{v_1}^{+\infty} \exp\left(\frac{1}{2} i\pi w^2\right) dw. \quad (14)$$

The diffraction field thus is computed as

$$A_1 = \frac{\int_{v_1}^{+\infty} \exp\left(i\pi \frac{w^2}{2}\right) dw}{\int_{-\infty}^{+\infty} \exp\left(i\pi \frac{w^2}{2}\right) dw} = \frac{1}{1+i} \int_{v_1}^{+\infty} \exp\left(i\pi \frac{w^2}{2}\right) dw \quad (15)$$

where we used the fact that

$$\int_{-\infty}^{+\infty} \exp\left(i\pi \frac{w^2}{2}\right) dw = 1 + i.$$

This is, in fact, the standard form of single knife-edge diffraction. Moreover, if  $h_0 = h_2 = 0$ , we obtain

$$\bar{v}_1 = h_1 \sqrt{\frac{2}{\lambda} \frac{r_1 + r_2}{r_1 r_2}}$$

that is, the Fresnel–Kirchhoff parameter (see [13], equation (3.26) in Chapter 3). We note further that the absolute value of  $A_1$  is identical to that obtained from the Vogler algorithm [1].

### B. Case Study 2: Double Knife-Edge Diffraction

When  $N = 2$ , following Theorem 1, the field  $u$  is given by

$$u(r_1 + r_2 + r_3, z_3) = \sqrt{\frac{1}{\lambda(r_1 + r_2 + r_3)}} \exp\left(-\frac{i\pi}{4}\right) \times \int_{-\infty}^{+\infty} u(r_1 + r_2, z_2) \exp\left(\frac{ik(z_3 - z_2)^2}{2r_3}\right) dz_2. \quad (16)$$

By using the result from  $N = 1$ , we obtain

$$u(r_1 + r_2 + r_3, z_3) = \text{const}_2 \int_{h_1}^{+\infty} \int_{h_2}^{+\infty} \exp\left(\frac{ik}{2} \left( \frac{(z_1 - h_0)^2}{r_1} + \frac{(z_2 - z_1)^2}{r_2} + \frac{(z_3 - z_2)^2}{r_3} \right)\right) dz_1 dz_2 \quad (17)$$

where

$$\text{const}_2 = \frac{1}{\lambda^{3/2}} \sqrt{\frac{1}{r_1(r_1 + r_2)(r_1 + r_2 + r_3)}} \exp\left(-i\frac{3\pi}{4}\right).$$

Thus, the diffraction field  $A_2$  is computed as

$$A_2 = \frac{\int_{h_1}^{+\infty} \int_{h_2}^{+\infty} \exp\left(\frac{ik}{2} d_2\right) dz_1 dz_2}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(\frac{ik}{2} d_2\right) dz_1 dz_2} \quad (18)$$

where

$$d_2 = \frac{(z_1 - h_0)^2}{r_1} + \frac{(z_2 - z_1)^2}{r_2} + \frac{(z_3 - z_2)^2}{r_3}.$$

We note that  $d_2$  can be rewritten as

$$d_2 = v_1^2 + v_2^2 + \frac{(h_3 - h_0)^2}{(r_1 + r_2 + r_3)}$$

where

$$v_1 = \frac{z_1 - h_0}{\mu_1} - \frac{\mu_1(z_2 - h_0)}{r_2} \\ v_2 = \frac{(z_2 - h_0)\eta}{\mu_2} - \frac{\mu_2(h_3 - h_0)}{\eta r_3}$$

with

$$\mu_1 = \sqrt{\frac{r_1 r_2}{r_1 + r_2}}, \quad \mu_2 = \sqrt{\frac{r_2 r_3}{r_2 + r_3}}, \quad \eta = \sqrt{\frac{r_2(r_1 + r_2 + r_3)}{(r_1 + r_2)(r_2 + r_3)}}.$$

By changing variables,

$$w_1 = \sqrt{\frac{2}{\lambda}} v_1 \\ w_2 = \sqrt{\frac{2}{\lambda}} v_2$$

we obtain

$$dw_1 = \sqrt{\frac{2}{\lambda}} \frac{1}{\mu_1} dz_1 \\ dw_2 = \sqrt{\frac{2}{\lambda}} \frac{\eta}{\mu_2} dz_2$$

and

$$w_2 \rightarrow q = \sqrt{\frac{2}{\lambda}} \left( \frac{(h_2 - h_0)\eta}{\mu_2} - \frac{\mu_2(h_3 - h_0)}{\eta r_3} \right)$$

when  $z_2 \rightarrow h_2$  as well as

$$w_1 \rightarrow p = \sqrt{\frac{2}{\lambda}} \left( \frac{h_1 - h_0}{\mu_1} - \frac{\mu_1(z_2 - h_0)}{r_2} \right)$$

when  $z_1 \rightarrow h_1$ . Thus, we can rewrite

$$E = \frac{\lambda}{2} \frac{\mu_1 \mu_2}{\eta} \exp\left(\frac{i\pi}{\lambda} \left( \frac{(h_3 - h_0)^2}{(r_1 + r_2 + r_3)} \right)\right) \times \int_p^{+\infty} \int_q^{+\infty} \exp\left(\frac{i\pi}{2} (w_1^2 + w_2^2)\right) dw_1 dw_2. \quad (19)$$

Thus,  $A_2$  can be computed as

$$A_2 = \frac{-i}{2} \int_p^{+\infty} \int_q^{+\infty} \exp\left(\frac{i\pi}{2} (w_1^2 + w_2^2)\right) dw_1 dw_2. \quad (20)$$

We note that a special case where  $h_0 = h_3 = 0$  is presented in [4]. This is consistent with the result for double knife-edge diffraction in [22].

#### IV. CONCLUSION

We have proved the relationship between the parabolic equation and the Vogler algorithm derivation for multiple knife-edge diffraction. By considering a relatively homogeneous propagation medium, we show that the standard parabolic equation can be simplified to the parabolic equation of diffraction theory. Then, we have proved that given a point (or punctual) source at the transmitter, solving this equation for multiple knife-edge models ( $N \geq 1$ ) is equivalent to the well-known Vogler algorithm derivation. Indeed, two different methods provide solutions to the same problem. The result of the established link between the two methods here is useful for cross-checking in practice and we could use the insight from the proof to develop new methods for specific problems.

#### APPENDIX

##### A. Proof of Lemma 1

*Proof:* We remind that [4] (see Chapter 2, Section II.4),

$$\mathcal{F}\left[\frac{\partial^2 u(x, z)}{\partial z^2}\right] = -4\pi^2 p^2 U(x, p), \quad \mathcal{F}\left[\frac{\partial u(x, z)}{\partial x}\right] = \frac{\partial U(x, p)}{\partial x}$$

where  $\mathcal{F}$  is the Fourier transform and  $U$  refers to the Fourier transform of the function  $u$  with respect to the variable  $z$ . By convention, the Fourier transform pair is defined as

$$U(x, p) = \int_{-\infty}^{+\infty} u(x, z) \exp(-2i\pi pz) dz, \quad u(x, z) = \int_{-\infty}^{+\infty} U(x, p) \exp(2i\pi zp) dp.$$

By transforming (3) to the frequency domain, we obtain

$$-4\pi^2 p^2 U(x, p) + 2ik \frac{\partial U}{\partial x}(x, p) = 0. \quad (21)$$

Then, the solution  $U(x, p)$  is given by

$$U(x, p) = \exp\left(-\frac{2i\pi^2 p^2}{k} x\right) U(0, p) \quad (22)$$

where we used, in the above, the fact that a solution of the first-order linear ordinary differential equation  $ay(x) + by'(x) = 0$  has the following form  $y(x) = \exp(-\frac{ax}{b})C_0$ , where  $C_0$  is a constant that depends on initial conditions. We note that since the product in the frequency domain  $p$  is equivalent to the convolution product in the

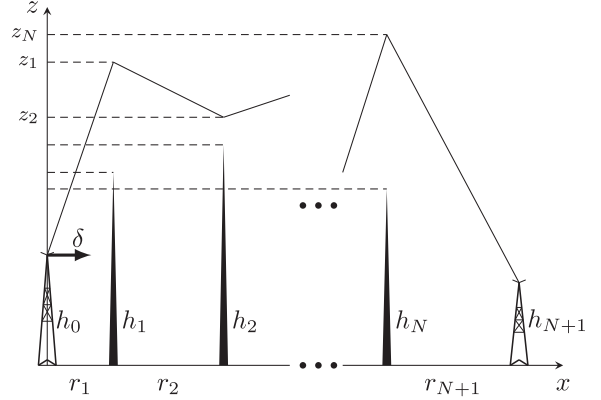


Fig. 2. Geometry of multiple knife edges with a point (or punctual) source  $\delta$  at the transmitter.

space domain  $z$ , we can evaluate  $u(x, z)$  from  $U(x, p)$  by using the inverse Fourier transform  $\mathcal{F}^{-1}$  and the following identity

$$\mathcal{F}^{-1}\left[\exp\left(-\frac{2i\pi^2 p^2 x}{k}\right)\right] = \sqrt{\frac{1}{\lambda x}} \exp\left(-\frac{i\pi}{4}\right) \exp\left(\frac{ikz^2}{2x}\right)$$

which can be proved by applying the integral of a Gaussian function:

$$\int_{-\infty}^{+\infty} \exp(-ax^2 + bx + c) dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right)$$

where  $a = ((2i\pi^2 x)/k)$ ,  $b = 2i\pi z$  and  $c = 0$ . We thus conclude the proof.  $\square$

##### B. Proof of Lemma 2

*Proof:* Following the geometry of multiple knife-edge diffraction and considering two consecutive knife edges  $m$  and  $m+1$ , the path length difference can be written as

$$d(z_m, z_{m+1}) = \sqrt{(z_{m+1} - z_m)^2 + r_{m+1}^2} - r_{m+1} = r_{m+1} \left( \sqrt{1 + \left(\frac{z_{m+1} - z_m}{r_{m+1}}\right)^2} - 1 \right) \approx \frac{(z_{m+1} - z_m)^2}{2r_{m+1}} \quad (23)$$

where we used the Maclaurin series of  $(1+x)^{1/2}$ . By summing the path length difference of all the segments for  $m = 0, \dots, N$  (by convention,  $z_0 = h_0$  and  $z_{N+1} = h_{N+1}$ ), we then achieve (5). We thus conclude the proof.  $\square$

##### C. Proof of Theorem 1

*Proof:* Let  $\{z_m\}_{m=1}^N$  be the height variables above  $\{h_m\}_{m=1}^N$ , respectively (see Fig. 2). We assume a point (or punctual) source at the transmitter, that is,  $u(x, z) = \delta(x)\delta(z - h_0)$ , where  $\delta$  is the Dirac delta function. Following Lemma 1, we compute the field  $u$  at a point  $(r_1, z_1)$ , where  $z_1 > h_1$ , as follows:

$$u(r_1, z_1) = \sqrt{\frac{1}{\lambda r_1}} \exp\left(-\frac{i\pi}{4}\right) \int_{-\infty}^{+\infty} u(0, v) \exp\left(\frac{ik(z_1 - v)^2}{2r_1}\right) dv = \sqrt{\frac{1}{\lambda r_1}} \exp\left(-\frac{i\pi}{4}\right) \exp\left(\frac{ik(z_1 - h_0)^2}{2r_1}\right) \quad (24)$$

where the second equality is obtained by using the translation property for the Dirac delta function. As a consequence, we can calculate the field  $u$  at a point  $(r_1 + r_2, z_2)$  as (see Fig. 2)

$$u(r_1 + r_2, z_2) = \frac{1}{\lambda \sqrt{r_1(r_1 + r_2)}} \exp\left(-\frac{i\pi}{2}\right) \times \int_{h_1}^{+\infty} \exp\left(\frac{ik}{2} \left(\frac{(z_1 - h_0)^2}{r_1} + \frac{(z_2 - z_1)^2}{r_2}\right)\right) dz_1. \quad (25)$$

The lower limit  $h_1$  is used here instead of  $-\infty$  since  $u = 0$  from  $-\infty$  to  $h_1$ . By applying this procedure successively, that is,  $u(r_1 + r_2 + r_3, z_3), \dots, u(\sum_{m=1}^{N+1} r_m, h_{N+1})$ , we obtain the desired result (6), where

$$\text{const}_N = \frac{1}{(\sqrt{\lambda})^{N+1} \sqrt{r_1(r_1 + r_2) \cdots (r_1 + \cdots + r_{N+1})}} \times \exp\left(-\frac{i\pi(N+1)}{4}\right) \quad (26)$$

is the constant related to  $u(\sum_{m=1}^{N+1} r_m, h_{N+1})$ . We note that since this constant will be canceled out when computing the ratio between  $u$  at the receiver and the corresponding free-space field, we do not define  $\text{const}_N$  in the statement of Theorem 1. We prove this statement by induction. We have already shown that it is true for  $N = 1, 2$  in Sections III-A and III-B. We assume that (6) holds for  $N - 1$ , that is,

$$u\left(\sum_{m=1}^N r_m, h_N\right) = \text{const}_{N-1} \int_{h_1}^{+\infty} \cdots \int_{h_{N-1}}^{+\infty} \exp(ikd_{N-1}) dz_1 \dots dz_{N-1}. \quad (27)$$

Again, by applying Lemma 1, we have

$$\begin{aligned} u\left(\sum_{m=1}^{N+1} r_m, h_{N+1}\right) &= \sqrt{\frac{1}{\lambda \left(\sum_{m=1}^N r_m\right)}} \exp\left(-\frac{i\pi}{4}\right) \\ &\times \int_{-\infty}^{+\infty} u\left(\sum_{m=1}^N r_m, z_N\right) \exp\left(\frac{ik(h_{N+1} - z_N)^2}{2h_{N+1}}\right) dz_N \\ &= \text{const}_N \int_{h_1}^{+\infty} \cdots \int_{h_N}^{+\infty} \exp(ikd_N) dz_1 \dots dz_N. \end{aligned} \quad (28)$$

Thus, the formula (6) holds by induction. Moreover, recall that  $A_N = (E/E_0)$  where the free-space field  $E_0$  can be computed by setting the lower limits of integral of  $u$  to  $-\infty$  (i.e.,  $h_{j=1}^N \rightarrow -\infty$ ), thus concluding the proof.  $\square$

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