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***Abstract*—In this paper, we consider problem of estimating diffraction attenuation from approximation of a terrain using multiple bridged knife-edge model. This model can be considered as a generalization of well-known multiple knife-edge one where spaces between knife-edge are bridged by reflecting surfaces. A series based-standard solution is presented in the literature but suffers from its very high computational complexity. We, thus, propose to generalize the recursive Vogler algorithm developed for multiple knife-edge model to tackle this problem. To reduce complexity, the proposed algorithm exploit the recursive form to avoid repeated calculations of the existing solution. Moreover, we provide a complexity analysis of both the standard algorithm and the proposed algorithm based on number of computed integrals. Both theoretical and numerical results show that our proposed algorithm is faster than the original one while having identical accuracy, hence proving the effectiveness of our solution.**

***Index Terms*—Multiple knife-edge diffraction, Vogler method, recursive algorithm, terrain and radio propagation modeling**

I. Introduction

Estimation of diffraction attenuation is an important problem used in radar [1], [2], propagation loss over irregular terrains [1], [3]–[5], wireless sensor network [6], and 5G wireless communication and beyond [7]. Considering, for example, a low altitude transmission among aeronautical systems and ground stations in mountain regions, diffraction plays one of dominant roles in mechanism due to the presence of terrain irregularities and obstacles. Accurate estimation of diffraction can, thus, provide advantages for channel modeling, radar coverage, or interference calculation and suppression, to name a few.

A. Related works

In the literature, depending on specific applications and their corresponding terrains of interest, we can model terrains by several basic geometric shapes such as single knife-edge [1], multiple knife-edges [8]–[11], cylinders [12]–[14], wedges [15]–[21] or combination of several basic shapes [22].

Different from those approaches, we are interested in the case where a terrain can be approximately represented by multiple bridged knife-edges model [23]. This model is formed by bridging spaces among knife-edges (see Fig. 1). Moreover, the model also considers ground reflection of each bridged segment and hence are suitable for characterizing both solid terrains and irregular terrains (i.e., in the case where number of knife-edges and segments are selected to be big enough).

The main advantage of this approach is its flexibility. We mean that, i) it includes the most simple and popular model, multiple knife-edges, as a particular case and takes into account ground reflection which is often omitted in some models; ii) it allows to obtain a good approximation of curved terrains or small radius hills by a series of segments. In applications such as prediction of diffraction over a spherical Earth [23] or in transition region between knife edge theory to smooth earth diffraction one [24], the later is essential.

In [25], the multiple bridged knife-edges model is introduced first in the framework of Fresnel-Kirchhoff theory for estimating multiple knife-edge diffraction of a solid terrain. The disadvantage of this method is, however, its numerical error of evaluating an integration to infinity of an oscillating function which is difficult to preserve its accuracy.

Later on, in [23], the author presents a connection between the derivation of [25] with the seminar work of Vogler [8]. A series solution, in a similar form as the Vogler algorithm, is also given for diffraction estimation. This solution overcomes the mentioned error and is accurate with confirmation from some standard tests. Still, its high computational complexity limits its potential applications in practice.

In terms of modeling, multiple knife-edge model combined with ground reflection is presented in [26] and [27]. It is shown that the overall attenuation is combination of field components related to a specific parameter of multiple knife-edge diffraction one. However, in a direct implementation, the number of component fields that the method needs to compute increases significantly to number of knife-edges (i.e., precisely, 2^N where N is number of knife-edges). Moreover, for approximation of curved terrains or hills, this approach can be inappropriate. Other approach for estimating diffraction attenuation in different edge setups is to use 2D transition function in the uniform geometrical theory of diffraction (UTD) (see [18], [28], [29] and references therein for more details).

B. Main contributions

To tackle the high computation complexity problem in [23], we propose a generalized recursive Vogler algorithm for multiple bridged knife-edge diffraction. The proposed algorithm takes advantage of the recursive form to avoid repeated calculation as the direct series solution, thus speeding up the

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TABLE I: Functions and symbols used throughout this paper.

Symbol/function	Definition
$\mathrm{i} \triangleq \sqrt{-1}$	Complex number
$I(m,\beta) \triangleq \frac{2}{m!\sqrt{\pi}} \int_{\beta}^{\infty} (u-\beta)^m \exp\left(-u^2\right) du$	Repeated integrals of the complementary error function [31]
$\binom{n}{m} \triangleq \frac{n!}{(n-m)!m!}$	Binomial coefficient
$k \triangleq \frac{2\pi}{\lambda}$	Wave number

calculation time. Moreover, we analyze in detail the computational complexity of four algorithms: our proposed algorithm, the series solution [23] as well as direct and recursive Vogler algorithms [8] in terms of number of computed integrals. The numerical results in several standard scenarios show that our proposed algorithm is much faster than the original series solution while having identical accuracy, thus proving the effectiveness of our analysis. The initial result of this study is reported in [30]. Here, we provide an improved algorithm and additional performance analysis and assessments.

The remainder of paper is organize as follow. In Section II, we provide the background and existing solution of multiple bridged knife-edge diffraction. Then, the proposed algorithm is described in Section III. We illustrate a special case of $N = 4$ in Section IV. In Section V, we provide a complexity analysis and comparison. Numerical results are given in Section VI. Finally, Section VII concludes the paper.

Notation: The functions and symbols used throughout this paper is presented in Table I.

II. The series based solution for multiple bridged knife-edge diffraction

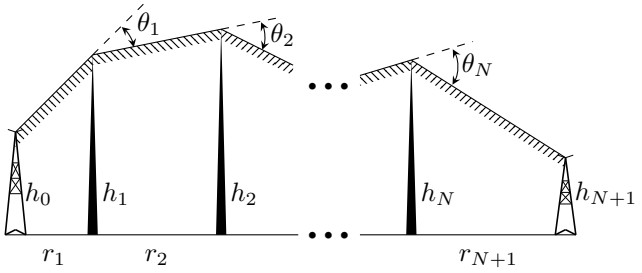

Fig. 1: Illustration of geometry for multiple bridged knife-edges.

In this section, we summarize the results of applying the Fresnel-Kirchhoff theory to multiple bridged knife-edge diffraction presented in [25] and [23] and assumptions used to derive such results.

An illustration of of geometry for multiple bridged knife-edges is shown in Fig. 1. We denote: i) h_0 and h_{N+1} are the transmitter and receiver heights respectively; ii) $\{h_n\}_{n=1}^{N}$ are the knife-edge heights to a reference surface; iii) $\{\theta_n\}_{n=1}^{N}$ are diffraction angles; and iv) $\{r_n\}_{n=1}^{N+1}$ are $N+1$ separation distances among knife-edges. Moreover, for sake of simplicity, the reflecting surfaces is assumed be perfect. It means that their reflection coefficients are -1. Following derivation from [25] and [23], the diffraction attention, A_N, for the multiple bridged knife-edge model is obtained by computing the following integral

$$\begin{aligned} A_N =& (1/\pi)^{N/2} K_N \exp\left(\sigma_N - \sigma'_N\right) \\ & \times \int_{\beta_1}^{\infty} \cdots \int_{\beta_N}^{\infty} \sum_{q_1=1-\lambda_1}^{\mu_1} \cdots \sum_{q_{N-1}=1-\lambda_{N-1}}^{\mu_{N-1}} (-1)^s \\ & \times \exp\left(2f\right) \exp\left(-\sum_{n=1}^{N} u_n^2\right) du_1 \cdots du_N \end{aligned} \tag{1}$$

where

$$K_N = \begin{cases} 1 \quad \text{for } N = 1 \\ \left[\frac{\left(\sum_{n=1}^{N+1} r_n\right) \prod_{n=1}^{N} r_n}{\prod_{n=1}^{N} \left(r_n + r_{n+1}\right)}\right]^{1/2} \quad , \ N \geq 2 \end{cases}, \tag{2}$$

$$\sigma_N = \sum_{n=1}^{N} \beta_n^2, \tag{3}$$

$$\sigma'_N = \mathrm{i}k \sum_{n=0}^{N} \frac{\left(h_{n+1} - h_n\right)^2}{2r_n}, \tag{4}$$

$$s = \sum_{n=1}^{N-1} q_n, \tag{5}$$

$$f = \begin{cases} 0 \quad \text{for } N = 1 \\ \sum_{n=1}^{N-1} \gamma_n \quad , \ N \geq 2, \end{cases} \tag{6}$$

$$\gamma_n = \alpha_n (-1)^{q_n} \left(u_n - \beta_n\right) \left(u_{n+1} - \beta_{n+1}\right) \tag{7}$$

$$\alpha_n = \left[\frac{r_n r_{n+1}}{\left(r_n + r_{n+1}\right)\left(r_{n+1} + r_{n+2}\right)}\right]^{1/2}, \ 1 \leq n \leq N-1, \tag{8}$$

$$\beta_n = \theta_n \left[\frac{\mathrm{i}k r_n r_{n+1}}{2\left(r_n + r_{n+1}\right)}\right]^{1/2}, \ 1 \leq n \leq N. \tag{9}$$

Here, λ_n and μ_n are set to one for the initial MBKED problem. If the nth reflecting bridge is omitted, we will assign μ_n to zero. Moreover, as mentioned in [23], λ_n will be set to zero occasionally if the Babinet's principle[1] [32] is applied. .

This result is obtained by adapting the Helmholtz integral to multiple bridged knife-edge diffraction for radio propagation [23]. In particular, the Helmholtz integral which represents the field E at a point $\mathbf{X}_{n+1} = (x_{n+1}, y_{n+1}, z_{n+1})^T$ is written as (see Eq. 1 in [25])

$$\begin{aligned} & E\left(\mathbf{X}_{n+1}\right) = \\ & \frac{1}{4\pi} \iint_S \left[E\left(\mathbf{X}_n\right) \frac{\partial}{\partial x_j} \left(\frac{\exp\left(\mathrm{i}kR\right)}{R}\right) - \frac{\exp\left(\mathrm{i}kR\right)}{R} \frac{\partial E\left(\mathbf{X}_n\right)}{\partial x}\right] dS \end{aligned} \tag{10}$$

where $E\left(\mathbf{X}_n\right)$ refers to the field at $\mathbf{X}_n = (x_n, y_n, z_n)^T$ on surface S, dS defines a surface element at point $\mathbf{Y}$, and R is the distance between $\mathbf{X}$ and $\mathbf{Y}$. The following assumptions (AS) are, then, used to make evaluation of the Helmholtz integral tractable [25]:

- *AS-1*: small wavelength approximation. It means that the wavelength is supposed to be much smaller than length

[1]Following the Babinet's principle, we can convert a diffraction problem involving thin diffracting screens into two new easier problems to solve.

of any segment of the considered propagation path. This assumption holds for very high frequency (VHF) and ultra high frequency (UHF) bands.
- *AS-2*: paraxial approximation. This assumption corresponds to the small angle approximation.
- *AS-3*: the regions of terrain perpendicular to the considered path is uniform. It means that the considered terrain is simplified from three dimensions to two ones. In our case, the direction along x-axis is propagation direction and the terrain height is along z-axis.
- *AS-4*: backscattered fields are not used in the Kirchhoff approximation.

By exploiting those assumptions and taking into account perfect reflecting surfaces, the field relative to free-space one is presented as in (1) after applying the result from evaluation of (10) repeatedly for N knife-edges.

A series based solution to compute (1) is proposed in [23] as follows

Theorem 1. *(Evaluation of A_N [23]): Given the following notation, $p_1 \equiv p, p_N \equiv 0, p_{N+1} \equiv 0, \alpha_N \equiv 1$,*

$$A_N = 2^{N'-N} K_N \sum_{p=0}^{\infty} 2^p \sum_{p_2=0}^{p_1=p} \cdots \sum_{p_{N-1}=0}^{p_{N-2}} \varepsilon \times \prod_{n=1}^{N} \frac{(p_{n-1}-p_{n+1})! \alpha_n^{p_n-p_{n+1}} I\left(p_{n-1}-p_{n+1}, \beta_n\right)}{(p_n-p_{n+1})!} \tag{11}$$

where N' is number of n values, $n = 1, \cdots, N$, for which $\lambda_n = \mu_n = 1$; the value of ε is given by the following condition: if there is at least one n so that $\lambda_n = \mu_n = 1$ and $p_n - p_{n+1}$ is even, then $\varepsilon = 0$. Otherwise, $\varepsilon = (-1)^q$ with q is the number of values of n for which $\lambda_n = 0$ and $p_n - p_{n+1}$ is even.

Even providing accurate result, the calculation of A_N as presented in Theorem 1 is not efficient in terms of computational complexity. To overcome this problem, we propose in next section a fast recursive algorithm which is inspired by the recursive Vogler algorithm introduced in [8] and proved later in [11]. We note that the series based algorithm following Theorem 1 and our proposed algorithm have a series based form. To distinguish two algorithms, we will use the term "recursive" for our algorithm to emphasize its key implementation characteristic.

III. Proposed method: Generalized recursive Vogler algorithm

Before presenting the main results, we first introduce the following lemma which is useful for algorithm derivation and analysis later.

Lemma 1. *(A recursive representation of multinomial theorem [11]) Following* (6)*, f^m can be given by* (12) *for $N \geq 3$ at the top of the next page*

By applying Lemma 1, we benefit from avoiding repeated calculation (see Section V for a detail analysis). We propose the following two main results:

Theorem 2. *(Recursive computation of A_N): Let*

$$C\left(N-1, m_{N-2}, m_{N-3}\right) = (-1)^{q_{N-1} m_{N-2}} \times (m_{N-3})! \alpha_{N-1}^{m_{N-2}} I\left(m_{N-3}, \beta_{N-1}\right) I\left(m_{N-2}, \beta_N\right). \tag{13}$$

Then, given the following notation

$$i = m_{N-L}, j = m_{N-L-1}, k = m_{N-L-2}$$
$$2 \leq L \leq N-2, N \geq 4$$

and the recursive relationship

$$C\left(N-L, j, k\right) = \sum_{i=0}^{j} (-1)^{q_{N-L}(j-i)} \times \frac{(k-i)!}{(j-i)!} \alpha_{N-L}^{j-i} I\left(k-i, \beta_{N-L}\right) C\left(N-L+1, i, j\right), \tag{14}$$

the diffraction attenuation of MBKE, A_N is given by

$$A_N = \frac{1}{2^N} K_N \exp\left(\sigma_N - \sigma'_N\right) \times \sum_{q_1=1-\lambda_1}^{\mu_1} \cdots \sum_{q_{N-1}=1-\lambda_{N-1}}^{\mu_{N-1}} (-1)^s \sum_{m=0}^{\infty} I_m$$

where I_m is computed recursively as follows

$$I_m = 2^m \sum_{m_1=0}^{m} (-1)^{q_1(m-m_1)} \times \alpha_1^{m-m_1} I\left(m-m_1, \beta_1\right) C\left(2, m_1, m\right). \tag{15}$$

Proof. See Appendix A □

Compared to the recursive Vogler algorithm [8], the proposed theorem here is more general with the appearance the sums over $\{q_n\}_{n=1}^N$ which takes into account reflecting surfaces. In the other words, the recursive Vogler algorithm [8] is a specific case of the proposed algorithm.

Now we present our case of interest where bridging terrains are presented among all the knife edges, but not on either side of them. This case corresponds to full multiple bridged knife-edge model as in Fig. 1. In terms of mathematical representation, the sums over $\{q_n\}_{n=1}^N$ in Theorem 2 are 'absorbed' through some factors and we obtain a 'compacter' form. In particular, we achieve a generalized result of the recursive Vogler algorithm for multiple knife-edge model where the main differences are the factors presenting the appearance of perfect reflecting surfaces (i.e., $1-(-1)^{m_{N-2}}$, $1-(-1)^{j-i}$, and $1-(-1)^{m-m_1}$ in (16), (17) and (18) respectively)

Theorem 3. *(Recursive computation of A_N where bridging terrains are presented among all the knife edges, but not on either side of them): Let*

$$C\left(N-1, m_{N-2}, m_{N-3}\right) = \left(1-(-1)^{m_{N-2}}\right) \times (m_{N-3})! \alpha_{N-1}^{m_{N-2}} I\left(m_{N-3}, \beta_{N-1}\right) I\left(m_{N-2}, \beta_N\right). \tag{16}$$

$$f^m = \underbrace{\sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \underbrace{\left[\sum_{m_2=0}^{m_1} \binom{m_1}{m_2} \gamma_2^{m_1-m_2} \left[\cdots \underbrace{\left[\sum_{m_{N-2}=0}^{m_{N-3}} \binom{m_{N-3}}{m_{N-2}} \gamma_{N-2}^{m_{N-3}-m_{N-2}} \gamma_{N-1}^{m_{N-2}} \right]}_{1^{\text{st}}\text{term}} \right] \right]}_{(N-3)^{\text{th}}\text{term}}}_{(N-2)^{\text{th}}\text{term}} \tag{12}$$

Then, given the following notation

$$\begin{aligned} &i = m_{N-L}, j = m_{N-L-1}, k = m_{N-L-2} \\ &2 \le L \le N-2, N \ge 4 \end{aligned}$$

and the recursive relationship

$$\begin{aligned} &C\left(N-L,j,k\right) \\ &= \sum_{i=0}^{j} \left(1-(-1)^{j-i}\right) \\ &\quad \times \frac{(k-i)!}{(j-i)!} \alpha_{N-L}^{j-i} I\left(k-i,\beta_{N-L}\right) C\left(N-L+1,i,j\right), \end{aligned} \tag{17}$$

the diffraction attenuation of MBKE, A_N is given by

$$A_N = \frac{1}{2^N} K_N \exp\left(\sigma_N - \sigma'_N\right) \sum_{m=0}^{\infty} I_m$$

where I_m is computed recursively as follows

$$\begin{aligned} I_m = 2^m &\sum_{m_1=0}^{m} \left(1-(-1)^{m-m_1}\right) \\ &\times \alpha_1^{m-m_1} I\left(m-m_1,\beta_1\right) C\left(2,m_1,m\right) \end{aligned} \tag{18}$$

Proof. See Appendix B □

Implementation details: For implementation, we provide the following details to reduce the number of computed integrals *significantly*. First, if $(m_i - m_{i+1})$ is even, then $\left(1-(-1)^{m_i-m_{i+1}}\right) = 0$. Thus, we do not need to compute the corresponding coefficients and save the overall complexity. Second, we note that the calculation of coefficients C in (17) in a recursive way include two steps: forward and backward. In the forward step, we verify if the term $\left(1-(-1)^{j-i}\right)$ of $C\left(N-L,j,k\right)$ is zero or nonzero. If it is zero, we omit and continue to the next step. Otherwise, the integral will not be computed yet until we continue to check through $C\left(N-L+1,i,j\right)$ at the backward step be nonzero. It means that the integrals will only be computed if both $\left(1-(-1)^{j-i}\right)$ and $C\left(N-L+1,i,j\right)$ are nonzero. In this way, there is no redundant computed integral. Moreover, we also apply this procedure to compute I_m in (18). By applying this procedure, we gain in terms of complexity considerably (see proof of Proposition 4 for a detailed analysis).

IV. Case study of four knife-edges ($N = 4$)

To illustrate key steps, we consider a case study of $N = 4$. Using Taylor series expansion of $\exp(2f)$, we can rewrite (1) as

$$\begin{aligned} A_4 = &\left(1/\pi\right)^2 K_4 \exp\left(\sigma_4 - \sigma'_4\right) \\ &\times \sum_{q_1=1-\lambda_1}^{\mu_1} \cdots \sum_{q_3=1-\lambda_3}^{\mu_3} \left(-1\right)^s \sum_{m=0}^{\infty} I_m \end{aligned} \tag{19}$$

where

$$\begin{aligned} I_m = &\left(\frac{2^m}{m!}\right) \left(\frac{2}{\sqrt{\pi}}\right)^4 \int_{\beta_1}^{\infty} \int_{\beta_2}^{\infty} \int_{\beta_3}^{\infty} \int_{\beta_4}^{\infty} \\ &\left(\sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \sum_{m_2=0}^{m_1} \binom{m_1}{m_2} \gamma_2^{m_1-m_2} {\gamma_3}^{m_2} \right) \\ &\times \exp\left(-\sum_{n=1}^{4} u_n^2\right) du_1 \cdots du_4. \end{aligned} \tag{20}$$

A. General case

Recall that, from (6) and (7), for $N = 4$,

$$\gamma_n = (-1)^{q_n} \alpha_n \left(u_n - \beta_n\right) \left(u_{n+1} - \beta_{n+1}\right), n = 1, 2, 3. \tag{21}$$

Substituting (21) into (20) yields

$$\begin{aligned} I_m = &\left(\frac{2^m}{m!}\right) \left(\frac{2}{\sqrt{\pi}}\right)^4 \int_{\beta_1}^{\infty} \int_{\beta_2}^{\infty} \int_{\beta_3}^{\infty} \int_{\beta_4}^{\infty} \\ &\Bigg(\sum_{m_1=0}^{m} \frac{m!}{(m-m_1)!m_1!} (-1)^{q_1(m-m_1)} \\ &\times \alpha_1^{m-m_1} (u_1-\beta_1)^{m-m_1} (u_2-\beta_2)^{m-m_1} \\ &\times \Bigg(\sum_{m_2=0}^{m_1} \frac{m_1!}{(m_1-m_2)!m_2!} (-1)^{q_2(m_1-m_2)} \alpha_2^{m_1-m_2} \\ &\times (u_2-\beta_2)^{m_1-m_2} (-1)^{q_3 m_2} \alpha_3^{m_2} (u_3-\beta_3)^{m_1} (u_4-\beta_4)^{m_2} \Bigg) \Bigg) \end{aligned} \tag{22}$$

Let

$$C\left(3,m_2,m_1\right) = (-1)^{q_3 m_2} m_1! \alpha_3^{m_2} I\left(m_1,\beta_3\right) I\left(m_2,\beta_4\right). \tag{23}$$

Then we can simplify (22) as

$$I_m = \left(\frac{2^m}{m!}\right)\left(\frac{2}{\sqrt{\pi}}\right)\int_{\beta_1}^{\infty}\sum_{m_1=0}^{m}\frac{m!}{(m-m_1)!}(-1)^{q_1(m-m_1)} \times \alpha_1^{m-m_1}(u_1-\beta_1)^{m-m_1}C\left(2,m_1,m\right)\exp\left(-u_1^2\right)du_1 \tag{24}$$

where

$$C\left(2,m_1,m\right) = \sum_{m_2=0}^{m_1}\frac{(m-m_2)!}{(m_1-m_2)!}(-1)^{q_2(m_1-m_2)} \times \alpha_2^{m_1-m_2}I\left(m-m_2,\beta_2\right)C\left(3,m_2,m_1\right) \tag{25}$$

Finally, we obtain

$$I_m = 2^m\sum_{m_1=0}^{m}\alpha_1^{m-m_1}(-1)^{q_1(m-m_1)} \times I\left(m-m_1,\beta_1\right)C\left(2,m_1,m\right). \tag{26}$$

B. Specific case: full multiple bridged knife edges

In this section, we present a specific case where bridging terrains are presented among the knife edges, but not on either side of them. This case of interest is equivalent to a 'full' multiple bridged knife edges and allows to simplify the final form of A_4. In particular, $q_i = 0$ and 1 for $i = 1, \cdots, 3$ and we can exchange the sum related to q_i to the coefficient $C\left(3,m_2,m_1\right)$. Thus, we can rewrite A_4 as follows

$$A_4 = \frac{1}{2^4}C_4\exp\left(\sigma_4-\sigma_4'\right)\sum_{m=0}^{\infty}I_m \tag{27}$$

where

$$I_m = 2^m\sum_{m_1=0}^{m}\alpha_1^{m-m_1}I\left(m-m_1,\beta_1\right)C\left(2,m_1,m\right) \tag{28}$$

$$C\left(2,m_1,m\right) = \sum_{m_2=0}^{m_1}\frac{(m-m_2)!}{(m_1-m_2)!} \times \alpha_2^{m_1-m_2}I\left(m-m_2,\beta_2\right)C\left(3,m_2,m_1\right)$$

$$C\left(3,m_2,m_1\right) = m_1!\alpha_3^{m_2}I\left(m_1,\beta_3\right)I\left(m_2,\beta_4\right) \times \sum_{q_1}\sum_{q_2}\sum_{q_3}\rho_{q_1,q_2,q_3} \tag{29}$$

with

$$\rho_{q_1,q_2,q_3} = (-1)^{\psi(q_1,q_2,q_3)} \tag{30}$$

and

$$\psi\left(q_1,q_2,q_3\right) = q_1\left(m-m_1+1\right)+q_2\left(m_1-m_2+1\right)+q_3\left(m_2+1\right). \tag{31}$$

Indeed, we grouped the coefficients related to q_i in (26), (25), and (23) to obtain ρ_{q_1,q_2,q_3}. Let

$$\Theta = \sum_{q_1}\sum_{q_2}\sum_{q_3}\rho_{q_1,q_2,q_3}. \tag{32}$$

Considering all combination of q_i, the sum of all corresponding ρ_{q_1,q_2,q_3} can be computed by

$$\begin{aligned}\Theta = &\left(-1\right)^{\psi(0,0,0)}+\left(-1\right)^{\psi(0,0,1)}+\left(-1\right)^{\psi(0,1,0)}+\left(-1\right)^{\psi(0,1,1)} \\ &+\left(-1\right)^{\psi(1,0,0)}+\left(-1\right)^{\psi(1,0,1)}+\left(-1\right)^{\psi(1,1,0)}+\left(-1\right)^{\psi(1,1,1)} \\ &= \left[1+\left(-1\right)^{m_1-m_2+1}\right]\left[1+\left(-1\right)^{m_2+1}\right] \\ &+\left(-1\right)^{m-m_1+1}\left[1+\left(-1\right)^{(m_1-m_2+1)}\right]\left[1+\left(-1\right)^{(m_2+1)}\right] \\ &= \left[1-\left(-1\right)^{m-m_1}\right]\left[1-\left(-1\right)^{m_1-m_2}\right]\left[1-\left(-1\right)^{m_2}\right].\end{aligned} \tag{33}$$

Then, by re-arranging those factors, we obtain the exact result of Theorem 3 for $N = 4$

$$I_m = 2^m\sum_{m_1=0}^{m}\left(1-\left(-1\right)^{m-m_1}\right) \times \alpha_1^{m-m_1}I\left(m-m_1,\beta_1\right)C\left(2,m_1,m\right) \tag{34}$$

where

$$C\left(2,m_1,m\right) = \sum_{m_2=0}^{m_1}\left(1-\left(-1\right)^{m_1-m_2}\right) \times \frac{(m-m_2)!}{(m_1-m_2)!}\alpha_2^{m_1-m_2}I\left(m-m_2,\beta_2\right)C\left(3,m_2,m_1\right) \tag{35}$$

and

$$C\left(3,m_2,m_1\right) = \left(1-\left(-1\right)^{m_2}\right)(m_1)!\alpha_3^{m_2}I\left(m_{N-3},\beta_3\right)I\left(m_2,\beta_4\right). \tag{36}$$

Moreover, by applying the procedure as described in the implementation details, we gain number of computed integrals significantly.

V. Complexity Analysis and Comparison

In this section, we use number of repeated integrals of the complementary error function computed, referred here as η_I^N, as a complexity index for comparison. The motivation behind using this index is that it contributes the dominant part and correlates well with overall complexity of algorithms.

We provide an analysis of four algorithms: Vogler and recursive Vogler algorithms [8], the series based algorithm [23] and our proposed algorithm. Four algorithms have a similar form and can be considered as a class.

Let M be a truncated value of the index m. The complexity of the Vogler algorithm [8] is given by the following proposition.

Proposition 1. *Number of repeated integrals of the complementary error function of the Vogler algorithm [8] is given by*

$$\eta_I^N = N\prod_{n=1}^{N-1}\frac{(M+n)}{n}. \tag{37}$$

Proof. See Appendix C. □

The complexity of the recursive Vogler algorithm is presented as follows.

TABLE II: Complexity comparison in terms of computed integrals among algorithms for multiple knife-edge (MKE) and mutiple bridged knife-edge (MBKE) models.

MKE	Vogler algorithm [8]	$\eta_I^N = N\prod_{n=1}^{N-1}\frac{(M+n)}{n}$
	Recursive Vogler algorithm [8]	$\eta_I^N = 3\prod_{n=1}^{N-1}\frac{(M+n)}{n} + \prod_{n=1}^{N-2}\frac{(M+n)}{n} + \cdots + \prod_{n=1}^{2}\frac{(M+n)}{n}$
MBKE	Series based-algorithm [23]	if N is even, $\eta_I^N = \begin{cases} N\prod_{n=1}^{N-1}\frac{(M-1)+(N-1)-2(n-1)}{2n}, & \text{if } M \text{ is even} \\ N\prod_{n=1}^{N-1}\frac{M+(N-1)-2(n-1)}{2n}, & \text{otherwise} \end{cases}$ if N is odd, $\eta_I^N = \begin{cases} N\prod_{n=1}^{N-1}\frac{M+(N-1)-2(n-1)}{2n}, & \text{if } M \text{ is even} \\ N\prod_{n=1}^{N-1}\frac{(M-1)+(N-1)-2(n-1)}{2n}, & \text{otherwise} \end{cases}$
	Generalized Recursive Vogler algorithm	Upper bounded by η_I^N of the series based algorithm.

Proposition 2. *Following the recursive implementation in [8], number of repeated integrals of the complementary error function is given by*

$$\eta_I^N = 3\prod_{n=1}^{N-1}\frac{(M+n)}{n} + \prod_{n=1}^{N-2}\frac{(M+n)}{n} + \cdots + \prod_{n=1}^{2}\frac{(M+n)}{n}. \tag{38}$$

Proof. See Appendix D. □

The complexity of the series based-algorithm is given by the following proposition.

Proposition 3. *Following Theorem 1, number of repeated integrals of the complementary error function of the series based algorithm is given by*

- *If N is even,*

$$\eta_I^N = \begin{cases} N\prod_{n=1}^{N-1}\frac{(M-1)+(N-1)-2(n-1)}{2n}, & \textit{if } M \textit{ is even} \\ N\prod_{n=1}^{N-1}\frac{M+(N-1)-2(n-1)}{2n}, & \textit{otherwise} \end{cases} \tag{39}$$

- *If N is odd,*

$$\eta_I^N = \begin{cases} N\prod_{n=1}^{N-1}\frac{M+(N-1)-2(n-1)}{2n}, & \textit{if } M \textit{ is even} \\ N\prod_{n=1}^{N-1}\frac{(M-1)+(N-1)-2(n-1)}{2n}, & \textit{otherwise} \end{cases} \tag{40}$$

Proof. See Appendix E. □

Now, we present the complexity of our proposed algorithm.

Proposition 4. *Following the recursive implementation in Theorem 3, the number of repeated integrals of the complementary error function is upper bounded by that of the series based-algorithm.*

Proof. See Appendix F. □

It is worth mentioning that, by comparing the result from Proposition 1 and 2, for $N = 3$, η_I^3 of the Vogler algorithm and the recursive Vogler algorithm are identical in the case of the multiple knife edge (MKE) model; by comparing the result from Proposition 3 and 4, η_I^3 of the series based algorithm and the proposed algorithm are the same in the case of multiple bridged knife edge (MBKE) model; If we compare between the cases of MKE and MBKE, algorithms in MKE has higher complexity than those in MBKE.

For $N > 3$, the recursive form presents its advantage where for both MKE and MBKE, the recursive algorithms have lower value of η_I^N than their non-recursive ones. The gain in terms of complexity become significantly when M increases (i.e., the accuracy increase). For MBKE model, by providing an upper bound, the complexity of our proposed algorithm is always lower than or equal to that of the series based-algorithm, thus proving the effectiveness of the proposed solution (see also Fig. 2 for an illustration).

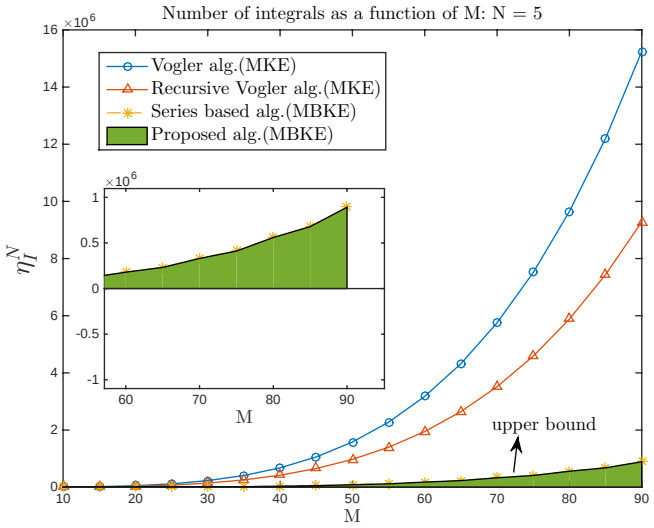

Fig. 2: Computational complexity comparison: number of computed integrals η_I^N as a function of the truncated value M. The lower curve is, the lower complexity is. The figure on the bottom left corner is a zoomed version of the main figure at the upper bound. The complexity of our proposed algorithm is lower than that of series based one.

VI. Numerical results

We provide numerical results to assess the effectiveness of our proposed algorithm. We compare our algorithm with series based algorithm in [23] and the recursive Vogler algorithms

in [8] in terms of accuracy and complexity. For the complexity, we provide the running time of CPU results as an additional and relative assessment besides theoretical analysis in Section V. To this end, we use several standard scenarios suggested in the literature [1], [8], [23], [27].

A. *Effect of truncated value M on accuracy*

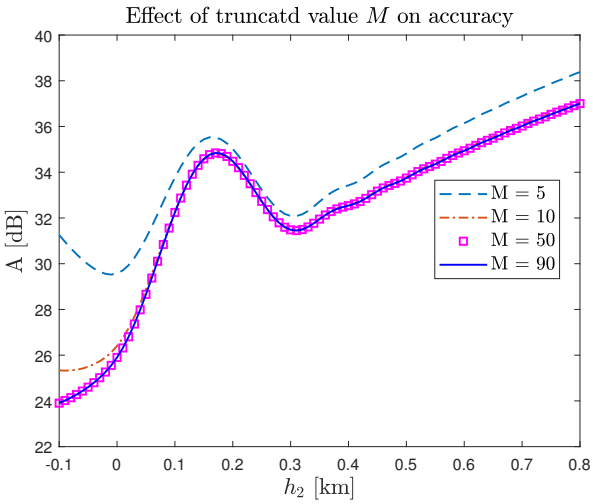

Fig. 3: Illustration of effect of truncated value M on accuracy. Four different values $M = 5, 10, 50, 90$ are chosen. The accuracy is proportional to values of M. When M is large enough (i.e., $M = 50$) in this experiment, increasing M has negligible errors. Here, the accuracy of final results corresponding to $M = 50$ and 90 are almost identical.

The truncated value M of index m has impact on both accuracy and computational complexity. The effect of M on the complexity of algorithms has already shown in Table II and Fig. 2. Thus, before comparing algorithms, we illustrate the effect of M on the accuracy of final result (see Fig. 3) in the following experiment: we study a propagation path with a distance of 30 km where there are two fixed knife-edges, h_1 and h_3, at distances of 10 km and 20 km from the transmitter respectively (see also Fig. 4 for this setup). A knife-edge with variable height, h_2, is placed at middle of two fixed knife-edge. Bridging terrain is presented between h_1 and h_2 as well as h_2 and h_3. The operation frequency is at 100 MHz. The transmitter and receiver are replaced in the reference plane (i.e., $h_0 = h_4 = 0$). Two fixed knife-edges have the same height of 100 m (i.e., $h_1 = h_3 = 100$m). When h_2 increases, the diffraction attenuation curves converges toward the curve of a single knife edge for multiple knife-edges and a wedge-shape terrain for multiple bridged knife-edges respectively.

We implement our proposed algorithm as described in Theorem 3. Here, we choose M from 5 to 90. We can see that the larger value of M is, the more precise result we obtain. For our tests presented later, we found by experiments that $M = 90$ is a good balance between accuracy and running time.

B. *Multiple knife-edges vs multiple bridged knife-edges*

We consider two experiments for the cases of $N = 3$ and $N = 5$ knife-edges respectively.

1) $N = 3$: In the first experiment, we use the same setup for $N = 3$ as described in the previous section. Since there exists two reflecting surfaces, the attenuation of multiple bridged knife-edges is larger than that of multiple edges. Moreover, the interference from two reflecting surfaces makes the corresponding attenuation curve fluctuate smaller than the curve with effect from two knife-edges. In terms of accuracy, as we can see in Fig. 4, the proposed algorithm is as accurate as the series based algorithm. In terms of speed (see Fig. 5), the proposed algorithm and the series based algorithm have similar running times and are faster than the recursive Vogler algorithm. This observation is correlation with our complexity analysis in Section V where values of η_I^N of the proposed algorithm and the series based algorithm are identical and smaller than and the recursive Vogler algorithm.

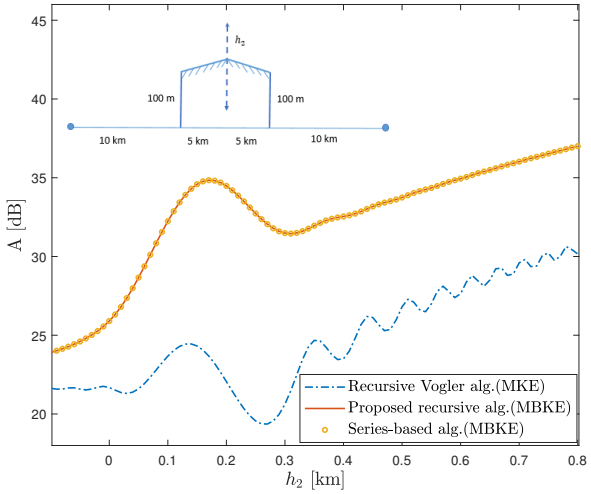

Fig. 4: Illustration on the accuracy for $N = 3$. Our recursive algorithm is as good as the series based algorithm. In this scenario, the diffraction attenuation from multiple bridged knife-edge model is significantly different from that of multiple knife-edge model. Thus, choosing a suitable model is essential.

2) $N = 5$: In the second experiment, we study a propagation path with a distance of 60 km (see Fig. 6). The transmitter and receiver heights are of 100 m above the reference plane. Three fixed knife-edges, h_1, h_3, and h_5 at distances of 10 km, 30 km and 50 km from the transmitter respectively are fixed. They have the same heights and are of 100 m (i.e., $h_1 = h_3 = h_5 = 100$ m). At the exact middle of two fixed knife-edges, there are two knife-edges with variable heights, h_2 and h_4 . Bridging terrains are presented between knife-edges but not either side of them. The operation frequency is at 100 MHz. Two configurations are examined: The former is referred to as 'h_4 only' in Fig. 6 where h_2 is fixed at 100 m below the reference plane (i.e., -100 m) and h_4 varies; The latter is referred to as h_2, h_4 where both h_2 and h_4 vary. We can observe in those configurations that, for accuracy, the proposed algorithm is, again, as accurate as the series based algorithm in both cases. However, in terms of speed (see Fig. 7, our proposed algorithm is faster than the series based, proving the effectiveness of proposed solution and analysis. Here, we

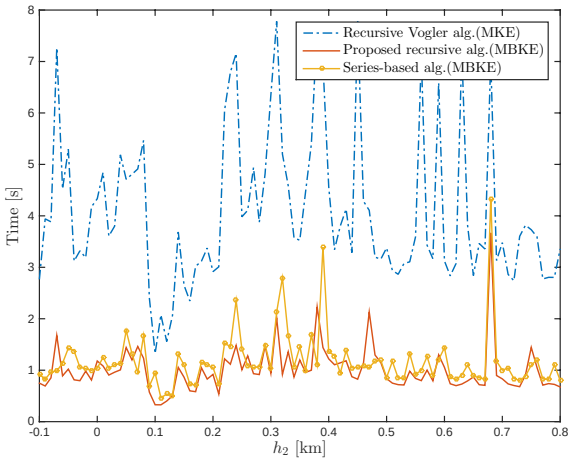

Fig. 5: Illustration on running time for $N = 3$. In this scenario, the proposed algorithm and the series based algorithm have similar running times and are faster than the recursive Vogler algorithm. This trend is consistent with our complexity analysis.

report the case where both h_2 and h_4 change. We confirm that a similar trend is observed for the case only h_4 changes. Compared to the recursive Volger algorithm, the running time of our proposed algorithm and the series based algorithm are lower because of smaller number of computed integrals. This is consistent with our analysis in Section V.

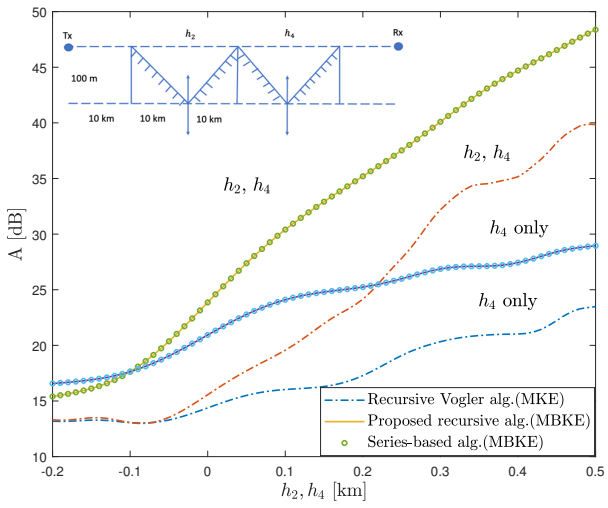

Fig. 6: Illustration on the accuracy for $N = 5$. In this scenario, again, our recursive algorithm has identical accuracy as the series based in both cases: i) h_4 changes and h_2 is fixed at -100 m; ii) both h_2 and h_4 change.

VII. Conclusion

In this paper, we have proposed a generalized recursive Vogler algorithm for multiple bridged knife-edge diffraction. The proposed algorithm is fast and accurate comparing to the existing solution. Its effectiveness is confirmed by both theoretical complexity analysis and numerical results. In a near future work, we will adapt a deep learning approach [33]

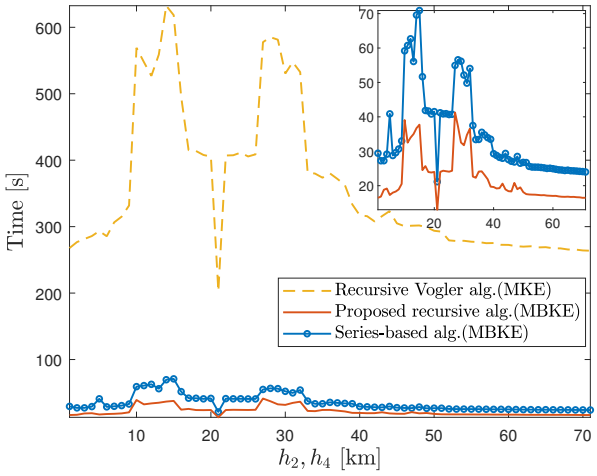

Fig. 7: Illustration on running time for $N = 5$, where both h_2 and h_4 change. The figure on the top right corner is a zoomed version of the main figure to compare proposed algorithm and the series based algorithm. Our proposed algorithm is faster than the series based algorithm. Moreover, both algorithms are much faster than the recursive Vogler agorithm.

proposed for multiple knife-edges diffraction to obtain a trade-off between accuracy and running time, aiming for real-time applications.

Appendix

A. Proof of Theorem 2

We provide a proof of Theorem 2 which is a generalization of the result in [11] for multiple knife edge diffraction. The proof can be outlined as follows. First, we transform (1) into a series by using Taylor series expansion of $\exp(2f)$ and Lemma 1. Then, we define a set of integrals, denoted as $\{\mathcal{I}_n\}_{n=1}^{N-2}$, that will provide a convenient way to link the result of the series from the previous step with the computation result of this set. Finally, we provide a formula representing the link and thus prove the Theorem 2. The proof thus can be divided into three steps:

- *Step 1*: We show the result of computing $\mathcal{I}_1$.
- *Step 2*: We provide a general expression for computing $\mathcal{I}_n$ and prove it by induction.
- *Step 3*: We show the relationship between $\mathcal{I}_n$ and I_m and conclude the proof.

By using Taylor series expansion of $\exp(2f)$, we can rewrite (1) as

$$A_N = \frac{1}{2^N} K_N \exp\left(\sigma_N - \sigma'_N\right) \times \sum_{q_1=1-\lambda_1}^{\mu_1} \cdots \sum_{q_{N-1}=1-\lambda_{N-1}}^{\mu_{N-1}} (-1)^s \sum_{m=0}^{\infty} I_m \tag{41}$$

where

$$I_m \triangleq \frac{2^m}{m!} \left(\frac{2}{\sqrt{\pi}}\right)^N \int_{\beta_1}^{\infty} \cdots \int_{\beta_N}^{\infty} f^m \exp\left(-\sum_{n=1}^{N} u_n^2\right) du_1 \cdots du_N \tag{42}$$

Furthermore, we want to compute N integrals of one-fold instead of one integral N-fold. To this end, we can expand f^m by applying Lemma 1. For convenience, we define the following set of integrals $\{\mathcal{I}_n\}_{n=1}^{N-2}$ related to each term of f^m after moving the integral to the desired term:

$$\mathcal{I}_n \triangleq \left(\frac{2}{\sqrt{\pi}}\right)^{n+1} \int_{\beta_{N-n}}^{\infty} \cdots \int_{\beta_N}^{\infty} \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \begin{pmatrix} m_{N-n-2} \\ m_{N-n-1} \end{pmatrix} \times \gamma_{N-n-1}^{m_{N-n-2}-m_{N-n-1}} \left(\sum_{i=N-n}^{N-1} \gamma_i\right)^{m_{N-n-1}} \times \exp\left(-\sum_{i=N-n}^{N} u_i^2\right) du_{N-n} \cdots du_N \tag{43}$$

Next, we show how we compute those integrals and link them to computation of I_m.

Step 1: We show the result of computing $\mathcal{I}_1$.

Following (43) , $\mathcal{I}_1$ is defined as

$$\mathcal{I}_1 = \left(\frac{2}{\sqrt{\pi}}\right)^2 \int_{\beta_{N-1}}^{\infty} \int_{\beta_N}^{\infty} \sum_{m_{N-2}=0}^{m_{N-3}} \binom{m_{N-3}}{m_{N-2}} \times \gamma_{N-2}^{m_{N-3}-m_{N-2}} \gamma_{N-1}^{m_{N-2}} \exp\left(-\left(u_{N-1}^2 + u_N^2\right)\right) du_{N-1} du_N \tag{44}$$

By substituting

$$\begin{aligned} &\gamma_{N-2}^{m_{N-3}-m_{N-2}} \gamma_{N-1}^{m_{N-2}} = \\ &(-1)^{q_{N-2}(m_{N-3}-m_{N-2})} \alpha_{N-2}^{m_{N-3}-m_{N-2}} \\ &\times (u_{N-2} - \beta_{N-2})^{m_{N-3}-m_{N-2}} (-1)^{q_{N-1} m_{N-2}} \\ &\times \alpha_{N-1}^{m_{N-2}} (u_{N-1} - \beta_{N-1})^{m_{N-3}} (u_N - \beta_N)^{m_{N-2}} \end{aligned} \tag{45}$$

into (44), we obtain the following result

$$\begin{aligned} \mathcal{I}_1 = &\sum_{m_{N-2}=0}^{m_{N-3}} (-1)^{q_{N-2}(m_{N-3}-m_{N-2})} \frac{(m_{N-3})!}{(m_{N-3}-m_{N-2})!} \\ &\times \alpha_{N-2}^{m_{N-3}-m_{N-2}} (u_{N-2} - \beta_{N-2})^{m_{N-3}-m_{N-2}} \\ &\times C\left(N-1, m_{N-2}, m_{N-3}\right) \end{aligned} \tag{46}$$

where

$$\begin{aligned} &C\left(N-1, m_{N-2}, m_{N-3}\right) = \\ &(-1)^{q_{N-1} m_{N-2}} \left(m_{N-3}\right)! \alpha_{N-1}^{m_{N-2}} I\left(m_{N-3}, \beta_{N-1}\right) I\left(m_{N-2}, \beta_N\right) \end{aligned} \tag{47}$$

Step 2: We provide a general expression for computing $\mathcal{I}_n$ and prove it by induction.

We introduce the following proposition used later in the proof of the main result and next steps

Proposition 5. *(A relationship between $\mathcal{I}_{n-1}$ and $\mathcal{I}_n$) The following relationship holds for $\mathcal{I}_{n-1}$ and $\mathcal{I}_n$, $2 \leq n \leq N-2$:*

$$\begin{aligned} \mathcal{I}_n = &\frac{2}{\sqrt{\pi}} \int_{\beta_{N-n}}^{\infty} \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \begin{pmatrix} m_{N-n-2} \\ m_{N-n-1} \end{pmatrix} \\ &\times \gamma_{m_{N-n-1}}^{m_{N-n-2}-m_{N-n-1}} \mathcal{I}_{n-1} \exp\left(-u_{N-n}^2\right) du_{N-n} \end{aligned} \tag{48}$$

Proof. From the definition (43), we can re-arrange $\mathcal{I}_n$ as

$$\begin{aligned} \mathcal{I}_n = &\frac{2}{\sqrt{\pi}} \int_{\beta_{N-n}}^{\infty} \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \begin{pmatrix} m_{N-n-2} \\ m_{N-n-1} \end{pmatrix} \gamma_{N-n-1}^{m_{N-n-2}-m_{N-n-1}} \\ &\times \left[\left(\frac{2}{\sqrt{\pi}}\right)^n \int_{\beta_{N-n+1}}^{\infty} \cdots \int_{\beta_N}^{\infty} \left(\sum_{i=N-n}^{N-1} \gamma_i\right)^{m_{N-n-1}} \right. \\ &\left. \exp\left(-\sum_{i=N-n+1}^{N} u_i^2\right) du_{N-n-1} \cdots du_N\right] \exp\left(u_{N-n}^2\right) du_{N-n} \\ \overset{(a)}{=} &\frac{2}{\sqrt{\pi}} \int_{\beta_{N-n}}^{\infty} \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \begin{pmatrix} m_{N-n-2} \\ m_{N-n-1} \end{pmatrix} \times \gamma_{m_{N-n-1}}^{m_{N-n-2}-m_{N-n-1}} \\ &\times \mathcal{I}_{n-1} \exp\left(-u_{N-n}^2\right) du_{N-n} \end{aligned} \tag{49}$$

where the equality (a) is achieved by expanding $\left(\sum_{i=N-n}^{N-1} \gamma_i\right)^{m_{N-n-1}}$ with Lemma 1. We thus conclude the proof. □

The result of computing the integral $\mathcal{I}_n$ can be obtained by the following proposition

Proposition 6. *For $2 \leq n \leq N-2$, $\mathcal{I}_n$ is given by*

$$\begin{aligned} \mathcal{I}_n = &\sum_{m_{N-n-1}=0}^{m_{N-n-2}} (-1)^{q_{N-n-1}(m_{N-n-2}-m_{N-n-1})} \\ &\times \frac{(m_{N-n-2})!}{(m_{N-n-2}-m_{N-n-1})!} \alpha_{N-n-1}^{m_{N-n-2}-m_{N-n-1}} \\ &\times \left(u_{N-n-1} - \beta_{N-n-1}\right)^{m_{N-n-2}-m_{N-n-1}} \\ &\times C\left(N-n, m_{N-n-1}, m_{N-n-2}\right) \end{aligned} \tag{50}$$

where

$$\begin{aligned} &C\left(N-n, m_{N-n-1}, m_{N-n-2}\right) = \\ &\sum_{m_{N-n}=0}^{m_{N-n-1}} (-1)^{q_{N-n}(m_{N-n-1}-m_{N-n})} \frac{(m_{N-n-2}-m_{N-n})!}{(m_{N-n-1}-m_{N-n})!} \\ &\times \alpha_{N-n}^{m_{N-n-1}-m_{N-n}} I\left(m_{N-n-2}-m_{N-n}, \beta_{N-n}\right) \\ &\times C\left(N-n+1, m_{N-n}, m_{N-n-1}\right). \end{aligned} \tag{51}$$

Proof. We prove this proposition by induction. First, we show that the result holds for $\mathcal{I}_2$. We then assume that the result (51) is true. Finally, we prove that it also holds for $\mathcal{I}_{n+1}$.

Following Lemma 5, $\mathcal{I}_2$ is given by

$$\begin{aligned} \mathcal{I}_2 = &\frac{2}{\sqrt{\pi}} \int_{\beta_{N-2}}^{\infty} \sum_{m_{N-3}=0}^{m_{N-4}} \begin{pmatrix} m_{N-4} \\ m_{N-3} \end{pmatrix} \gamma_{N-3}^{m_{N-4}-m_{N-3}} \\ &\times \mathcal{I}_1 \exp\left(-u_{N-2}^2\right) du_{N-2} \end{aligned} \tag{52}$$

We then substitute

$$\begin{aligned} &\gamma_{N-3}^{m_{N-4}-m_{N-3}} = \\ &(-1)^{q_{N-3}(m_{N-4}-m_{N-3})} \alpha_{N-3}^{m_{N-4}-m_{N-3}} \\ &\times \left(u_{N-3} - \beta_{N-3}\right)^{m_{N-4}-m_{N-3}} \left(u_{N-2} - \beta_{N-2}\right)^{m_{N-4}-m_{N-3}} \end{aligned} \tag{53}$$

and calculate the integral to achieve

$$\mathcal{I}_2 = \sum_{m_{N-3}=0}^{m_{N-4}} (-1)^{q_{N-3}(m_{N-4}-m_{N-3})} \frac{(m_{N-4})!}{(m_{N-4}-m_{N-3})!} \times \alpha_{N-3}^{m_{N-4}-m_{N-3}} (u_{N-3}-\beta_{N-3})^{m_{N-4}-m_{N-3}} \times C(N-2, m_{N-3}, m_{N-4}) \quad (54)$$

where

$$C(N-2, m_{N-3}, m_{N-4}) = \sum_{m_{N-2}=0}^{m_{N-3}} (-1)^{q_{N-2}(m_{N-3}-m_{N-2})} \frac{(m_{N-4}-m_{N-2})!}{(m_{N-3}-m_{N-2})!} \times \alpha_{N-2}^{m_{N-3}-m_{N-2}} I(m_{N-4}-m_{N-2}, \beta_{N-2}) \times C(N-1, m_{N-2}, m_{N-3}) \quad (55)$$

In a similar way as for $\mathcal{I}_2$, we can reach $\mathcal{I}_{n+1}$ by substituting

$$\gamma_{m_{N-n-2}}^{m_{N-n-3}-m_{N-n-2}} = (-1)^{q_{m_{N-n-2}}(m_{N-n-3}-m_{N-n-2})} \times \alpha_{m_{N-n-2}}^{m_{N-n-3}-m_{N-n-2}} \left(u_{m_{N-n-2}} - \beta_{m_{N-n-2}}\right)^{m_{N-n-3}-m_{N-n-2}} \times \left(u_{m_{N-n-1}} - \beta_{m_{N-n-1}}\right)^{m_{N-n-3}-m_{N-n-2}} \quad (56)$$

into (48) (i.e., we replace index n by $n+1$) as follows

$$\begin{aligned}
\mathcal{I}_{n+1} &= \frac{2}{\sqrt{\pi}} \int_{\beta_{N-n-1}}^{\infty} \sum_{m_{N-n-2}=0}^{m_{N-n-3}} \binom{m_{N-n-3}}{m_{N-n-2}} \\
&\gamma_{m_{N-n-2}}^{m_{N-n-3}-m_{N-n-2}} \mathcal{I}_n \exp\left(-u_{N-n-1}^2\right) du_{N-n-1} \\
&= \sum_{m_{N-n-2}=0}^{m_{N-n-3}} (-1)^{q_{m_{N-n-2}}(m_{N-n-3}-m_{N-n-2})} \\
&\times \frac{(m_{N-n-3})!}{(m_{N-n-3}-m_{N-n-2})!\,(m_{N-n-2})!} \alpha_{N-n-2}^{m_{N-n-3}-m_{N-n-2}} \\
&\times \left(u_{N-n-2} - \beta_{N-n-2}\right)^{m_{N-n-3}-m_{N-n-2}} \\
&\times \sum_{m_{N-n-1}=0}^{m_{N-n-2}} (-1)^{q_{N-n-1}(m_{N-n-2}-m_{N-n-1})} \\
&\frac{(m_{N-n-2})!}{(m_{N-n-2}-m_{N-n-1})!} \alpha_{N-n-1}^{m_{N-n-2}-m_{N-n-1}} \\
&\times C(N-n, m_{N-n-1}, m_{N-n-2}) \\
&\times \left[\frac{2}{\sqrt{\pi}} \int_{\beta_{N-n-1}}^{\infty} \left(u_{N-n-1} - \beta_{N-n-1}\right)^{m_{N-n-3}-m_{N-n-1}} \right. \\
&\left. \exp\left(-u_{N-n-1}^2\right) du_{N-n-1}\right] \\
&= \sum_{m_{N-n-2}=0}^{m_{N-n-3}} (-1)^{q_{m_{N-n-2}}(m_{N-n-3}-m_{N-n-2})} \\
&\times \frac{(m_{N-n-3})!}{(m_{N-n-3}-m_{N-n-2})!} \alpha_{N-n-2}^{m_{N-n-3}-m_{N-n-2}} \\
&\times \left(u_{N-n-2} - \beta_{N-n-2}\right)^{m_{N-n-3}-m_{N-n-2}} \\
&\times C(N-n-1, m_{N-n-2}, m_{N-n-3}) \quad (57)
\end{aligned}$$

where

$$\begin{aligned}
&C(N-n-1, m_{N-n-2}, m_{N-n-3}) \\
&= \sum_{m_{N-n-1}=0}^{m_{N-n-2}} (-1)^{q_{N-n-1}(m_{N-n-2}-m_{N-n-1})} \\
&\times \frac{(m_{N-n-3}-m_{N-n-1})!}{(m_{N-n-2}-m_{N-n-1})!} \alpha_{N-n-1}^{m_{N-n-2}-m_{N-n-1}} \\
&\times I(m_{N-n-3}-m_{N-n-1}, \beta_{N-n-1}) \\
&\times C(N-n, m_{N-n-1}, m_{N-n-2}) \quad (58)
\end{aligned}$$

It is straightforward to verify that the result for $\mathcal{I}_{n+1}$ is true from the statement of Proposition 6. We thus conclude the proof. □

Based on (47), (55), and (58), we then obtain the following corollary from Proposition 6 .

Corollary 1. *(The recursive relationship of C) Let*

$$\begin{aligned}
&m_0 = m, \\
&i = m_{N-L}, j = m_{N-L-1}, k = m_{N-L-2} \\
&2 \le L \le N-2, N \ge 4.
\end{aligned}$$

The following expression holds

$$\begin{aligned}
&C(N-L, j, k) \\
&= \sum_{i=0}^{j} (-1)^{q_{N-L}(j-i)} \\
&\times \frac{(k-i)!}{(j-i)!} \alpha_{N-L}^{j-i} I(k-i, \beta_{N-L})\, C(N-L+1, i, j).
\end{aligned}$$

Step 3: We show the relationship between $\mathcal{I}_n$ and I_m and conclude the proof.
The relationship between $\mathcal{I}_n$ and I_m is given by the following proposition

Proposition 7. *(A relationship between $\mathcal{I}_{N-2}$ and I_m) The following relationship holds*

$$I_m = \left(\frac{2^m}{m!}\right) \frac{2}{\sqrt{\pi}} \int_{\beta_1}^{\infty} \mathcal{I}_{N-2} \exp\left(-u_1^2\right) du_1. \quad (59)$$

Proof. Recall that, from the definition of (43), $\mathcal{I}_{N-2}$ is given by

$$\begin{aligned}
&\mathcal{I}_{N-2} = \\
&\left(\frac{2}{\sqrt{\pi}}\right)^{N-1} \int_{\beta_2}^{\infty} \cdots \int_{\beta_N}^{\infty} \sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \left(\sum_{i=2}^{N-1} \gamma_i\right)^{m_1} \\
&\times \exp\left(-\sum_{i=2}^{N} u_i^2\right) du_2 \cdots du_N \quad (60)
\end{aligned}$$

Moreover, we can represent an expansion of f^m by using the binomial theorem as

$$f^m = \sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \left(\sum_{i=2}^{N-1} \gamma_i\right)^{m_1} \quad (61)$$

By substituting (61) into (42) and rearranging the integrals corresponding to the suitable variables, we obtain

$$\begin{aligned} I_m &= \left(\frac{2^m}{m!}\right)\frac{2}{\sqrt{\pi}}\int_{\beta_1}^{\infty} du_1 \\ &\times\left[\left(\frac{2}{\sqrt{\pi}}\right)^{N-1}\int_{\beta_2}^{\infty} du_2\cdots\int_{\beta_N}^{\infty} du_N f^m \exp\left(-\sum_{n=2}^{N} u_n^2\right)\right] \\ &\times \exp\left(-u_1^2\right) du_1 \\ &= \left(\frac{2^m}{m!}\right)\frac{2}{\sqrt{\pi}}\int_{\beta_1}^{\infty}\mathcal{I}_{N-2}\exp\left(-u_1^2\right) du_1 \end{aligned} \tag{62}$$

We thus conclude the proof. □

Moreover, recall from Proposition 6 that

$$\begin{aligned} \mathcal{I}_{N-2} &= \sum_{m_1=0}^{m}(-1)^{q_1(m-m_1)}\frac{(m)!}{(m-m_1)!} \\ &\times \alpha_1^{m-m_1}(u_1-\beta_1)^{m-m_1} C\left(2,m_1,m\right) \end{aligned} \tag{63}$$

Finally, by substituting (63) into (59) of Proposition 7, I_m is given by

$$\begin{aligned} &I_m = \\ &\left(\frac{2^m}{m!}\right)\sum_{m_1=0}^{m}(-1)^{q_1(m-m_1)}\frac{(m)!}{(m-m_1)!}\alpha_1^{m-m_1} \\ &\times C\left(2,m_1,m\right)\left[\frac{2}{\sqrt{\pi}}\int_{\beta_1}^{\infty}(u_1-\beta_1)^{m-m_1}\exp\left(-u_1^2\right) du_1\right] \\ &= 2^m\sum_{m_1=0}^{m}(-1)^{q_1(m-m_1)}\alpha_1^{m-m_1} I\left(m-m_1,\beta_1\right) C\left(2,m_1,m\right) \end{aligned} \tag{64}$$

which is exact result of Theorem 2. We thus conclude the proof.

B. Proof of Theorem 3

Theorem 3 is a special case of Theorem 2 where the sum over $q_i, i=1,\cdots,N-1$ can be factorized into simple terms as presented in the following proposition.

Proposition 8. *Let*

$$\Omega_{q_1,\cdots,q_{N-1}} \triangleq (-1)^{\sum_{i=1}^{N-1} q_i(m_{i-1}-m_i+1)}, N>2 \tag{65}$$

Given the following notation,

$$m_0 = m, m_N = 0,$$

we obtain

$$\begin{aligned} \Theta_N &\triangleq \sum_{q_1=0}^{1}\cdots\sum_{q_{N-1}=0}^{1}\Omega_{q_1,\cdots,q_{N-1}} \\ &= \prod_{i=1}^{N-1}\left(1-(-1)^{m_{i-1}-m_i}\right). \end{aligned} \tag{66}$$

Proof. We also prove this lemma by induction. For $N=3$ and 4, the result of lemma holds (see Section IV-B). We assume that the lemma is true for $N=n$ and prove it true for $N=n+1$.

For $N=n+1$, we can compute Θ_{n+1} as

$$\begin{aligned} &\Theta_{n+1} = \\ &\sum_{q_1=0}^{1}\sum_{q_2=0}^{1}\cdots\sum_{q_n=0}^{1}(-1)^{\sum_{i=1}^{n} q_i(m_{i-1}-m_i+1)} \\ &= \sum_{q_1=0}^{1}\left[\sum_{q_2=0}^{1}\cdots\sum_{q_n=0}^{1}(-1)^{q_1(m-m_1+1)+\sum_{i=2}^{n} q_i(m_{i-1}-m_i+1)}\right] \\ &= \sum_{q_1=0}^{1}(-1)^{q_1(m-m_1+1)} \\ &\times\left[\sum_{q_2=0}^{1}\cdots\sum_{q_n=0}^{1}(-1)^{\sum_{i=2}^{n} q_i(m_{i-1}-m_i+1)}\right] \\ &\overset{(a)}{=} \sum_{q_1=0}^{1}(-1)^{q_1(m-m_1+1)}\prod_{i=2}^{n}\left(1-(-1)^{m_{i-1}-m_i}\right) \\ &= \prod_{i=1}^{n}\left(1-(-1)^{m_{i-1}-m_i}\right) \end{aligned} \tag{67}$$

$$= \prod_{i=1}^{n}\left(1-(-1)^{m_{i-1}-m_i}\right) \tag{68}$$

where the equality (a) is reached by using (66). We thus conclude the proof. □

Finally, by applying Proposition 8, it is straightforward to obtain the Theorem 3 by re-arranging factors from Theorem 2.

C. Proof of Proposition 1

We prove this proposition by induction. We first consider the Vogler algorithm. Recall that M is a truncated value of the index m. For $N \geq 3$, a general solution for computing multiple knife-edge A_N^{MKE} is given by [8]

$$A_N^{\text{MKE}} = \frac{1}{2^N} C_N \exp\left(\sigma_N\right)\sum\nolimits_{m=0}^{\infty} I_m \tag{69}$$

where

$$\begin{aligned} &I_m = \\ &2^m\sum_{m_1=0}^{m}\cdots\sum_{m_{N-2}}^{m_{N-3}}\prod_{i=1}^{N}\frac{(m_{i-1}-m_{i+1})!}{(m_i-m_{i+1})!}\alpha_i^{m_{i-1}-m_i} I\left(n_i,\beta_i\right) \end{aligned} \tag{70}$$

with, by using notation $m_0 = m$,

$$n_i = \begin{cases} m_0 - m_1 & i=1 \\ m_{i-2}-m_i & 2\leq i\leq N-1 \\ m_{N-2}-m_{N-1} & i=N \end{cases}. \tag{71}$$

For $N=3$, number of integrals inside summation is given by

$$\begin{aligned} \eta_I^3 &= 3\sum_{m=0}^{M}(m+1) \\ &= 3\frac{(M+1)(M+2)}{1\times 2} \end{aligned} \tag{72}$$

For $N=4$, we apply the result from $N=3$ and thus obtain

$$\eta_I^4 = 4\sum_{m=0}^{M}\frac{(m+1)(m+2)}{2} \tag{73}$$

To evaluate η_I^4, we build a telescoping sum as follows. We note that

$$\frac{(m+1)(m+2)}{2} = \frac{1}{6}\left[(m+1)(m+2)(m+3) - m(m+1)(m+2)\right]. \tag{74}$$

Thus, we can rewrite (73) as

$$\begin{aligned}\eta_I^4 &= 4\sum_{m=0}^{M}\frac{(m+1)(m+2)}{2}\\ &= \frac{4}{6}\sum_{m=0}^{M}\left[(m+1)(m+2)(m+3) - m(m+1)(m+2)\right]\\ &= 4\frac{(M+1)(M+2)(M+3)}{1\times 2\times 3}.\end{aligned} \tag{75}$$

where the last equality is obtain by canceling pairs of consecutive terms.

Now, we assume that, the expression of η_I^{N-1} is true. We show that it is also true for η_I^N

$$\eta_I^N = N\sum_{m=0}^{M}\prod_{n=1}^{N-2}\frac{(m+n)}{n} \tag{76}$$

In a similar way as for $N=4$, we note that

$$\prod_{n=1}^{N-2}\frac{(m+n)}{n} = \frac{1}{(N-1)!}\left[\prod_{n=1}^{N-1}(m+n) - \prod_{n=0}^{N-2}(m+n)\right] \tag{77}$$

By substituting (77) into (93), we obtain

$$\begin{aligned}\eta_I^N &= \frac{N}{(N-1)!}\sum_{m=0}^{M}\left[\prod_{n=1}^{N-1}(m+n) - \prod_{n=0}^{N-2}(m+n)\right]\\ &= N\prod_{n=1}^{N-1}\frac{(M+n)}{n}.\end{aligned} \tag{78}$$

We thus conclude the proof.

D. Proof of Proposition 2

The recursive algorithm for MKE has the same number of loops but the difference in terms of integrals arrangement, leading to different number of computed integrals. The proof is similar to that of Proposition 1. We illustrate this point for the case of $N=3$ and 4.

For $N=3$, following Theorem 3, we can write A_N explicitly as follows

$$A_N = \frac{1}{2^N}K_N\exp\left(\sigma_N - \sigma_N'\right)\sum_{m=0}^{M} I_m$$

where

$$\begin{aligned}I_m = 2^m\sum_{m_1=0}^{m}&\alpha_1^{m-m_1}m!\alpha_2^{m_1}\\ &\times I\left(m-m_1,\beta_1\right)I\left(m,\beta_2\right)I\left(m_1,\beta_3\right).\end{aligned} \tag{79}$$

Thus, it is clear that,

$$\eta_I^3 = 3\frac{(M+1)(M+2)}{2}. \tag{80}$$

For $N=4$, from (26), (25) and (23), after removing the related reflecting coefficients, we can express I_m as

$$\begin{aligned}I_m = 2^m\sum_{m_1=0}^{m}&\alpha_1^{m-m_1}\\ &\times I\left(m-m_1,\beta_1\right)\sum_{m_2=0}^{m_1}\frac{(m-m_2)!}{(m_1-m_2)!}\\ &\times\alpha_2^{m_1-m_2}m_1!\alpha_3^{m_2}\\ &\times I\left(m-m_2,\beta_2\right)I\left(m_1,\beta_3\right)I\left(m_2,\beta_4\right).\end{aligned} \tag{81}$$

We can count number of integrals as

$$\eta_I^4 = 3\prod_{n=1}^{3}\frac{(M+n)}{n} + \prod_{n=1}^{2}\frac{(M+n)}{n} \tag{82}$$

where the first term includes three times (i.e., $I\left(m-m_2,\beta_2\right), I\left(m_1,\beta_3\right), I\left(m_2,\beta_4\right)$) number of terms inside the loops and the second term includes number of integrals $I\left(m-m_1,\beta_1\right)$. From those cases, we can generalize for arbitrary $N>3$ by using the same method as the proof of Proposition 1 as

$$\eta_I^N = 3\prod_{n=1}^{N-1}\frac{(M+n)}{n} + \prod_{n=1}^{N-2}\frac{(M+n)}{n} + \cdots + \prod_{n=1}^{2}\frac{(M+n)}{n}. \tag{83}$$

We thus conclude the proof.

E. Proof of Proposition 3

Before providing the details, we note that, a necessary condition for $\epsilon\neq 0$ is that the last index of loops is odd. For example, for $N=3$ and 4 , their corresponding last indices, m_1 and m_2, are odd. To see this point, we consider the case of $N=3$ and 4 first.

For $N=3$, we have

$$\epsilon = \left(1-(-1)^{m-m_1}\right)\left(1-(-1)^{m_1}\right). \tag{84}$$

Observe that, we only need take into account the case of m is even and m_1 is odd (i.e., $\epsilon\neq 0$). We note further that, index m_1 runs from 0 to m. Thus, we obtain

- if the truncated value M is even, then

$$\begin{aligned}\eta_I^3 &= 3\sum_{\substack{m=0\\ m\ \text{even}}}^{M}\frac{m}{2}\\ &= 3\frac{(M+2)M}{(2\times 1)\times(2\times 2)}\end{aligned} \tag{85}$$

- if the truncated value M is odd, we have

$$\begin{aligned}\eta_I^3 &= 3\frac{\tilde{M}(\tilde{M}+2)}{(2\times 1)\times(2\times 2)}\\ &= 3\frac{(M+1)(M-1)}{(2\times 1)\times(2\times 2)}\end{aligned} \tag{86}$$

with $\tilde{M} = M - 1$. Since all items corresponding to odd index m are eliminated, number of integrals for index $m = M$ is zero.

For $N = 4$, we can observe that $\epsilon = (1-(-1)^{m-m_1})(1-(-1)^{m_1-m_2})(1-(-1)^{m_2}) \neq 0$ if the following condition is meet: m is *odd*, m_1 is even and m_2 is odd. By taking into account this condition, we use the formula for odd M from $N = 3$ as follows

$$\eta_I^4 = 4 \sum_{\substack{m=0 \\ m \text{ odd}}}^{M} \frac{(m+1)(m-1)}{8} \tag{87}$$

Let $m = 2n+1$. Then we can rewrite η_I^4 as

$$\eta_I^4 = \sum_{n=0}^{(M-1)/2} \frac{(n+1)n}{2} \tag{88}$$

In a similar way as in Proposition 1, we build a telescoping sum as follows. We note that

$$\frac{(n+1)n}{2} = \frac{1}{6}\left[(n+2)(n+1)n - (n+1)n(n-1)\right] \tag{89}$$

Then, we can calculate η_I^4 as

$$\eta_I^4 = 4 \sum_{n=0}^{(M-1)/2} \frac{(n+1)n}{2} \tag{90}$$

$$= \frac{4}{6} \sum_{n=0}^{(M-1)/2} \left[(n+2)(n+1)n - (n+1)n(n-1)\right]$$

$$= 4\frac{(M+3)(M+1)(M-1)}{(2\times 1)\times(2\times 2)\times(2\times 3)} \tag{91}$$

Again, by replacing M with $M-1$, we obtain the result of η_I^4 for M even as follows

$$\eta_I^4 = 4\frac{(M+2)M(M-2)}{(2\times 1)\times(2\times 2)\times(2\times 3)} \tag{92}$$

Now, we assume that, the expression of η_I^{N-1} is true. We show that it is also true for η_I^N. Without loss of the generality, we assume that N *is even and* M *is odd*. We use the formula from odd $N-1$ with odd M to compute η_I^N as follows

$$\eta_I^N = N \sum_{\substack{m=0 \\ m \text{ odd}}}^{M} \prod_{i=1}^{N-2} \frac{(m-1)+(N-2)-2(i-1)}{2i} \tag{93}$$

Let $m = 2n+1$. Then we obtain η_I^N as

$$\eta_I^N = N \sum_{n=0}^{(M-1)/2} \left[\prod_{i=1}^{N-2} \frac{2n+(N-2)-2(i-1)}{2i}\right] \tag{94}$$

To build a telescoping sum, the expression inside the square brackets can be represented as

$$\prod_{i=1}^{N-2} \frac{2n+(N-2)-2(i-1)}{2i} =$$

$$\frac{1}{2^{N-1}(N-1)!}\Bigg[\prod_{i=0}^{N-2}(2n+(N-2)-2(i-1)) - \prod_{i=1}^{N-1}(2n+(N-2)-2(i-1))\Bigg] \tag{95}$$

Thus, we achieve the following result

$$\eta_I^N = \frac{N}{2^{N-1}(N-1)!} \times \sum_{n=0}^{(M-1)/2}\Bigg[\prod_{i=0}^{N-2}(2n+(N-2)-2(i-1)) - \prod_{i=1}^{N-1}(2n+(N-2)-2(i-1))\Bigg]$$

$$= \frac{N}{2^{N-1}(N-1)!}\prod_{i=0}^{N-2}((M-1)+(N-2)-2(i-1)) \tag{96}$$

Let $n = i+1$. Then, after rearranging coefficients, we obtain the exact result of η_I^N as presented in Proposition 3 for even N and odd M

$$\eta_I^N = N\prod_{n=1}^{N-1}\frac{M+(N-1)-2(n-1)}{2n}. \tag{97}$$

By replacing M by $M-1$, we attain the exact result of η_I^N as presented in Proposition 3 for even N and even M

$$\eta_I^N = N\prod_{n=1}^{N-1}\frac{(M-1)+(N-1)-2(n-1)}{2n}. \tag{98}$$

In a similar way, we now prove the result for odd N and even and odd M.

Without loss of the generality, we assume that N *is odd and* M *is even*. We use the formula from even $N-1$ with even M to compute η_I^N as follows

$$\eta_I^N = N \sum_{\substack{m=0 \\ m \text{ even}}}^{M} \prod_{i=1}^{N-2} \frac{(m-1)+(N-2)-2(i-1)}{2i} \tag{99}$$

Let $m = 2n$. Then we obtain η_I^N as

$$\eta_I^N = N \sum_{n=0}^{M/2} \left[\prod_{i=1}^{N-2} \frac{(2n-1)+(N-2)-2(i-1)}{2i}\right] \tag{100}$$

To build a telescoping sum, we represent the expression inside the square brackets as

$$\prod_{i=1}^{N-2} \frac{(2n-1)+(N-2)-2(i-1)}{2i} =$$

$$\frac{1}{2^{N-1}(N-1)!}\Bigg[\prod_{i=0}^{N-2}((2n-1)+(N-2)-2(i-1)) - \prod_{i=1}^{N-1}((2n-1)+(N-2)-2(i-1))\Bigg] \tag{101}$$

Thus, we achieve the exact result for odd N and even M

$$\begin{aligned}\eta_I^N &= \frac{N}{2^{N-1}(N-1)!} \\ &\times \sum_{n=0}^{M/2}\Bigg[\prod_{i=0}^{N-2}\left((2n-1)+(N-2)-2(i-1)\right) \\ &\qquad -\prod_{i=1}^{N-1}\left((2n-1)+(N-2)-2(i-1)\right)\Bigg] \\ &= \frac{N}{2^{N-1}(N-1)!}\prod_{i=0}^{N-2}\left((M-1)+(N-2)-2(i-1)\right) \\ &= N\prod_{n=1}^{N-1}\frac{M+(N-1)-2(n-1)}{2n}. \end{aligned} \tag{102}$$

Again, by replacing M by $M-1$, we attain the exact result of η_I^N for odd N and odd M

$$\eta_I^N = N\prod_{n=1}^{N-1}\frac{(M-1)+(N-1)-2(n-1)}{2n}. \tag{103}$$

We thus conclude the proof.

F. Proof of Proposition 4

Recall that, for MBKE of $N=3$

$$\begin{aligned} I_m &= 2^m\sum_{m_1=0}^{m}\left(1-(-1)^{m-m_1}\right) \\ &\quad\times \alpha_1^{m-m_1} I(m-m_1,\beta_1)\left(1-(-1)^{m_1}\right) \\ &\quad\times m!\alpha_2^{m_1} I(m,\beta_2) I(m_1,\beta_3). \end{aligned} \tag{104}$$

Thus, the number of integrals η_I^3 is identical to the case of Proposition 3. For $N=4$, from (26), (25) and (23), we can express I_m as

$$\begin{aligned} I_m &= 2^m\sum_{m_1=0}^{m}\left(1-(-1)^{m-m_1}\right) \\ &\quad\times \alpha_1^{m-m_1} I(m-m_1,\beta_1) C(2,m_1,m) \end{aligned} \tag{105}$$

where

$$\begin{aligned} C(2,m_1,m) &= \sum_{m_2=0}^{m_1}\left(1-(-1)^{m_1-m_2}\right) \\ &\times \frac{(m-m_2)!}{(m_1-m_2)!}\alpha_2^{m_1-m_2} I(m-m_2,\beta_2) C(3,m_2,m_1) \end{aligned} \tag{106}$$

and

$$\begin{aligned} &C(3,m_2,m_1) \\ &= \left(1-(-1)^{m_2}\right)(m_1)!\alpha_3^{m_2} I(m_{N-3},\beta_3) I(m_2,\beta_4). \end{aligned} \tag{107}$$

Our aim now is to show that η_I^4 for the proposed algorithm is less than or equal to that of the series based algorithm. To simplify analysis and without loss of generality, we simplify the expression for computing A in the proposed algorithm (refer to as A_4^{pro}) and the series based algorithm (refer to as A_4^{ser}) respectively by keeping only integrals and their related factors as presented in (108) and (109).

$$\begin{aligned} &A_4^{\text{ser}} = \\ &\sum_{m=0}^{M}\sum_{m_1=0}^{m}\sum_{m_2=0}^{m_1}\left(1-(-1)^{m-m_1}\right)\left(1-(-1)^{m_1-m_2}\right)\left(1-(-1)^{m_2}\right) \\ &\times I(m-m_1,\beta_1) I(m-m_2,\beta_2) I(m_{N-3},\beta_3) I(m_2,\beta_4) \end{aligned} \tag{108}$$

$$\begin{aligned} &A_4^{\text{pro}} = \\ &\sum_{m=0}^{M}\sum_{m_1=0}^{m}\left(1-(-1)^{m-m_1}\right) I(m-m_1,\beta_1)\sum_{m_2=0}^{m_1}\left(1-(-1)^{m_1-m_2}\right)\left(1-(-1)^{m_2}\right) \\ &\times I(m-m_2,\beta_2) I(m_{N-3},\beta_3) I(m_2,\beta_4) \end{aligned} \tag{109}$$

Several values of M and the corresponding indices (m,m_1,m_2) to count the number of computed integrals η_I^4 for two algorithms is given in Table III. We use this table to analyze the computational differences between two algorithms. First, we observe that there is no difference in terms of η_I^4 in the case of two successive values of M, for example $M=1,2$ or $M=3,4$. This is due to the fact that we count the number of integrals (for this case) if m_2 is odd AND m_1 is even AND m is odd. Second, let us consider the difference between two algorithms appearing in the case of $M=5,6$. The series based-algorithm computes 4 integrals corresponding to 4 couples indices, resulting in $\eta_I^4=16$ in total (see Eq. (108) and Table III).

TABLE III: Illustration of indices (m,m_1,m_2) to count the number of computed integrals η_I^4. Underlined indices show the difference between number of computed integrals of the series based algorithm and proposed algorithm. The more M increases, the more significantly our proposed algorithm gains.

	$M=1,2$	$M=3,4$	$M=5,6$	$M=7,8$
(m,m_1,m_2)		$(3,2,1)$	$(3,2,1)$	$(3,2,1)$
			$(5,2,1)$	$(5,2,1)$
			$\underline{(5,4,1)}$	$\underline{(5,4,1)}$
			$\underline{(5,4,3)}$	$\underline{(5,4,3)}$
				$\underline{(7,4,1)}$
				$\underline{(7,4,3)}$
				$\underline{(7,6,1)}$
				$\underline{(7,6,3)}$
				$\underline{(7,6,5)}$
η_I^4(series based)	0	4	16	40
η_I^4(proposed)	0	4	15	36
Our gain	0	0	1	4

In contrast, our algorithm computes 8 integrals for the first two couple indices (i.e.,$(m,m_1,m_2)=(3,2,1)$ and $(m,m_1,m_2)=(5,2,1)$). For the last two couple indices (i.e.,$(5,4,1)$ and $(5,4,3)$), our algorithm, however, compute 1 integral corresponding to $(m,m_1)=(5,4)$ and 6 integrals insides the loops of $m_2=1$ and $m_2=3$, resulting in 15 in total (see Eq. (109) and Table III). Thus, we gain 1 computed integral. In a similar way, for $M=7,8$, we gain 4 computed integrals. When M increases, this gain increases *significantly*. In practice, we saw that, it increases exponentially to M.

Again, for $N>4$, the structure of problem is in the similar

way. Using the same argument, we can show that η_I^N of proposed algorithm is upper bounded by η_I^N of the series based- algorithm. We thus conclude the proof.

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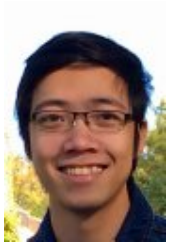

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