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Abstract

An evaluation of fatigue property determination using numerical simulation is presented for a variant of the staircase procedure with the reuse of unbroken specimens. The simulated staircases are analysed with the Maximum Likelihood Estimation method to obtain the fatigue parameters: the median fatigue limit and the coefficient of variation. For each condition, 1000 staircases are simulated and evaluated to obtain a statistical study. The gain on the fatigue scatter uncertainties with the variant procedure is highlighted. However, an assumption on the damage accumulation must be stated when reloading specimens up to their failure. Thus, staircases are simulated with damage consideration.

Keywords: Staircase method, Maximum likelihood estimation, Interval censoring, Reliability, Weibull model

1. Introduction

Engineering parts are designed to resist fatigue for a certain number of cycles N_{ref}. In the case of high-cycle fatigue, the design must take into account the statistical distribution of the fatigue properties of the material for this given number of cycles. Indeed, in order to obtain the best reliability, the parts are designed for the highest allowed probability of failure to attain the design life. The maximum allowable stress is determined with the statistical distribution of the fatigue properties of the material at N_{ref} cycles (schematically represented in Fig. 1a). The goal of fatigue test procedures is therefore to characterize this distribution, thus the median fatigue limit Σ_{50} and the associated scattering. The median S-N curve and its dispersion (dotted curves) are also schematically represented. However, during real fatigue tests, the probabilistic nature is on the number of cycles performed at given stress amplitudes Σ_0, set by the machine operator (Fig. 1b).

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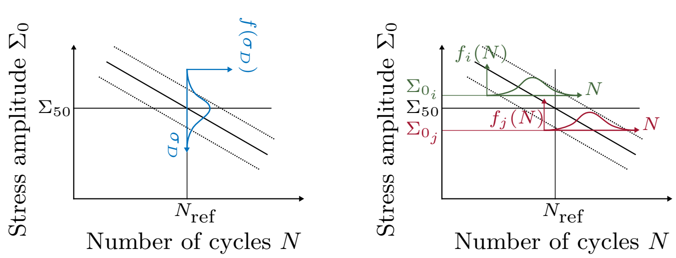
Several testing strategies can be used, and the staircase (or up-and-down) method is a very common approach. This procedure was introduced and statistically analysed by Dixon and Mood in 1948 [1]. The procedure and analysis are presented in various standards, such as the ASTM [2]. The protocol is simple: a specimen is tested at a given initial stress amplitude for the specified number of cycles N_{ref}. If the specimen survives (run-out), the stress amplitude is increased for the next specimen, likewise it is decreased if the specimen fails. After testing all the specimens, the Dixon and Mood equations can be applied to the resulting staircase in order to estimate the median fatigue limit and its standard deviation for the number of cycles N_{ref}. Because of its simplicity, the staircase procedure is mainly used in an industrial context. This method is efficient to estimate the median fatigue limit with few specimens. However the standard deviation is hardly estimated: the Dixon and Mood condition on the sample size (around 50 specimens for the determination of the standard deviation using the Dixon-Mood analysis) is rarely fulfilled. In order to increase the reliability of the parts, it is important to quantify the uncertainies of this analysis to optimise the fatigue property determination.

Several publications and investigations have been already conducted to quantify the uncertainties as to the standard deviation and to improve the reliability of its determination, especially for small sample sizes [3, 4]. More recently, numerical simulations have been performed in order to evaluate the staircase procedure with the Dixon-Mood analysis [5, 6, 7]. This method is a convenient approximation for the maximum likelihood estimation (MLE), applied to this particular staircase procedure with a normal distribution. Thus, some authors directly used the MLE in order to quantify the reliability of the staircase method in different configurations (*e.g.* influence of the step size or of the initial stress level) [7, 8, 9, 10]. Other studies focus on the comparison of evaluation techniques for the classic procedure [7, 11, 12] or on the development of new staircase procedures and evaluation techniques [10].

In this paper, a simulation-based investigation is performed to evaluate a new procedure with the reuse of unbroken specimens. The goals are to propose fatigue determination methods for reloaded specimens and to quantify the



(a) Distribution representing the fatigue properties of the specimen population

(b) Distributions of the number of cycles for stress amplitudes

Fig. 1. Schematic representation of the probabilistic nature of fatigue tests

Nomenclature

α	Scale parameter of the Weibull model
β	Shape parameter of the Weibull model
C_p	Load step coefficient (staircase procedure)
C_v	Coefficient of variation (material property)
C_{bd}	Slope coefficient used for the fatigue strength calculation (fatigue analysis procedure)
C_{bm}	Slope coefficient used for the staircase simulation (material property)
Σ_0	Stress amplitude during the procedure
Σ_{50}	Median fatigue limit
σ_D	Randomly selected fatigue strength
$\sigma_{N_{\text{ref}}}$	Allowable stress at the reference number of cycles
b_d	Slope of the S-N curve for the fatigue strength calculation (fatigue analysis procedure)
b_m	Slope of the S-N curve for the staircase simulation (material property)
F	Cumulative Distribution Function (CDF)
f	Probability Density Function (PDF)
m	Weibull modulus
n	Number of specimens in the sample
N_{ref}	Number of cycles defining the fatigue life
N_r	Number of cycles to failure
p	Load step in the staircase procedure

uncertainties of the fatigue distribution parameters with these methods. The evaluation of the accuracy of a fatigue analysis method is based on two considerations: the median response and the scatter in results. For each condition, 1000 staircases will be simulated and evaluated in order to obtain the distribution of the fatigue property parameters. The median responses and the scatters in results can be analysed for the median fatigue limit and for the fatigue scatter parameter, in relation to the procedure (number of specimens, step size...) and the fatigue determination method (censored data, fatigue strength calculation...).

The present paper is divided into five parts. The first part of this paper will focus on the generation of the simulated staircases and their analysis. The second part will focus on the influence of the number of specimens per sample on the fatigue property uncertainties. A comparison between the classic staircase procedure and the variant with reloaded specimens is then performed. The goal is to evaluate the possible reduction in uncertainties, as a function of the sample size. When reloading specimens, possible damage accumulation could affect the staircase proceedings. The third part presents the damage consideration in the staircase simulation in order to perform two different studies. Indeed, two methods are defined and used to analyse the staircases with reloaded specimens, one based on censored data (with interval censoring) and the other on the fatigue strength calculation of each specimen. Consequently, the fourth part presents a validity domain of the interval censoring method with damage accumulation. Indeed, this analysis does not take into account the cumulative damage, so a validity domain should be determined as a function of the material parameters (initial fatigue scatter and shape of the S-N curve) and the parameter used during the staircase procedure (step of the stress amplitude between two levels). And finally, the fifth part presents the second study on the sensitivity of the parameter used for the fatigue strength calculation of each specimen with damage consideration. Then, the different methods are compared and the possibility of reusing unbroken specimens is examined.

2. Staircase simulation and evaluation technique

For this simulation-based investigation, some hypotheses are required: the failure of the specimen is supposed to occur from only one failure mechanism, so the fatigue properties are described with only one distribution. Moreover, the distribution is assumed to follow a Weibull model, for the simulations and the evaluation techniques. Usually, this probability distribution is anticipated when analysing real fatigue tests, the exact shape of the distribution is hard to evaluate with a small number of specimens. Furthermore, during the simulation of the staircase, fatigue tests are perfect, meaning that the exact values of stress amplitude are virtually applied without any noise. With these assumptions, the uncertainties calculated from the 1000 simulated staircases are idealised. However, the values still allow the calculation of the possible gain, in perfect conditions, with the reuse of unbroken specimens in a staircase procedure.

Virtual specimens are generated. A specimen i is only represented by its fatigue strength σ_{D_i} at the reference number of cycles N_{ref}. The fatigue strengths are randomly selected from the statistical distribution of the fatigue properties represented using a Weibull distribution.

The two staircase procedures are then simulated. No cumulative damage is considered in the first part of the paper. So, the applied stress amplitude is directly compared to the fatigue strength of the specimen. If the applied stress amplitude is lower than the randomly selected fatigue strength, the specimen survives at the N_{ref} cycles. Likewise, a failure occurs if the applied stress amplitude is greater than the fatigue strength.

The simulated staircases are then evaluated using the Maximum Likelihood Estimation (MLE). It allows the determination of the fatigue parameters, depending on the hypothesis assumed for the fatigue analysis, thus with censored data or with calculated allowable stresses.

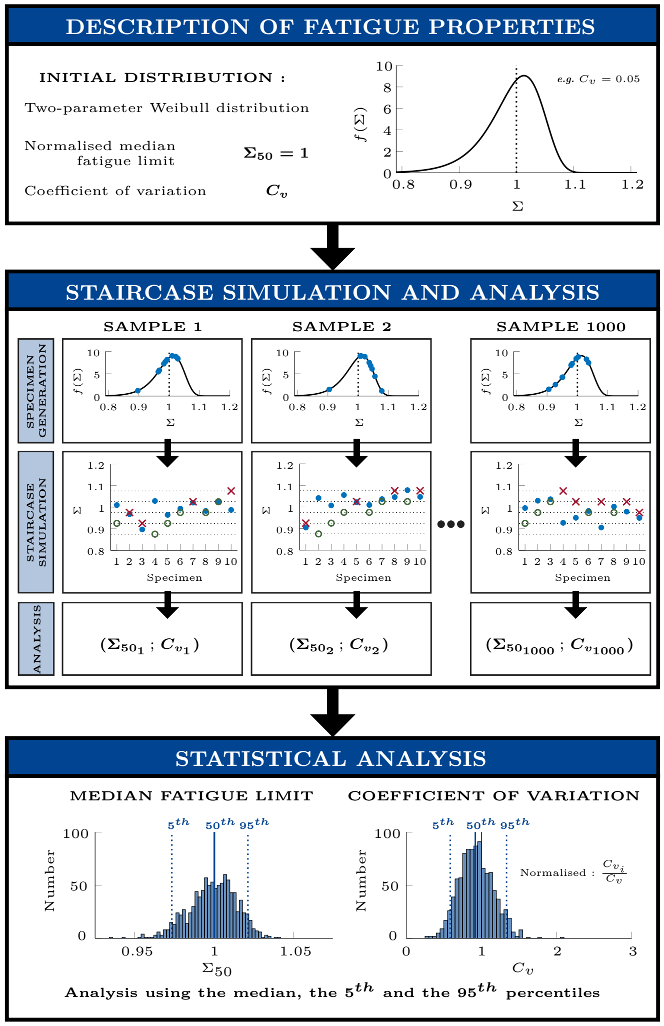


Fig. 2. Principle of the uncertainties evaluation of the fatigue distribution parameters - Example for the classic staircase procedure and $n = 10$ specimens per sample

2.1. Distribution of fatigue properties

With small samples, the choice of the distribution shape has little effect on the evaluation of the median endurance limit. However, when dealing with the fatigue scatter and the low failure probabilities, the tail of the probability density function is an important consideration and the choice of the distribution shape will strongly affect the results. With a large number of specimens, some statistical tests are possible in order to evaluate the shape of the distribution. However, during real fatigue test analysis, where there are often not enough data, the shape is usually assumed (from experience of other similar material or by choice).

One condition of the Dixon and Mood analysis is that the random variable must be normally distributed. This condition is not really restrictive: transformation could be applied and therefore other distribution shapes are possible and already studied. Müller [7] used a log-normal distribution in order to study different evaluation techniques of the staircase procedure (the MLE with censored data and the Dixon and Mood analysis for instance). Beaumont [12] simulates staircases and Locati procedures with normal, log-normal and Weibull distributions.

In this paper, the Weibull model [13] is chosen to describe the fatigue property distribution. This choice was motivated by other studies and more precisely by the fact that this model can naturally take into account the scale and the stress heterogeneity effect on the fatigue properties [14, 15]. Moreover, the Weibull distribution also appears in the self-heating curve modelling [16]. Furthermore, the Dixon and Mood analysis is not used in this paper, the MLE is directly used. The Weibull distribution will be used for the generation of the specimens and will also be assumed during the fatigue property determination.

A two-parameter Weibull distribution is used in this simulation-based investigation. The random variable x is the fatigue strength of the specimen. The Cumulative Distribution Function (CDF), which also represents the failure probability for a given stress amplitude, is obtained using

$$F(x) = 1 - \exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right), \tag{1}$$

where α is the scale parameter and β the shape parameter. The Probability Density Function (PDF) is then given by

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right). \tag{2}$$

The two parameters chosen to describe the fatigue properties are the median fatigue limit Σ_{50} and the coefficient of variation C_v, thus the standard deviation divided by the mean fatigue limit. The coefficient of variation is only dependent on the Weibull modulus m and is calculated using

$$\mathrm{C}_v = \frac{\sqrt{\Gamma\left(1+\frac{2}{m}\right) - \Gamma^2\left(1+\frac{1}{m}\right)}}{\Gamma\left(1+\frac{1}{m}\right)}. \tag{3}$$

The shape and scale parameters of the initial distribution are then calculated using

$$\alpha = \frac{\Sigma_{50}}{(\ln 2)^{1/m}} \tag{4}$$

and

$$\beta = m. \tag{5}$$

The initial distribution is thus described by the median fatigue limit Σ_{50} and the coefficient of variation C_v. From the distribution defined by these two parameters, the fatigue strengths of the specimens σ_{D_i} are randomly selected. The goal is therefore to generate a large number of staircases in order to obtain statistical distributions of these two fatigue parameters after the fatigue analysis of the staircases. Indeed, these obtained distributions allow the evaluation of the median response and the scatter in results, in order to quantify the uncertainties of the evaluation technique (Fig. 2).

2.2. Staircase procedures

In this section, the simulation of two staircase procedures is presented. No damage consideration is considered in the first part of the paper. Staircases are simulated from a sample of n generated specimens.

The choice of the load step between two stress levels is important when performing a staircase procedure. In this paper, a normalised coefficient is defined by

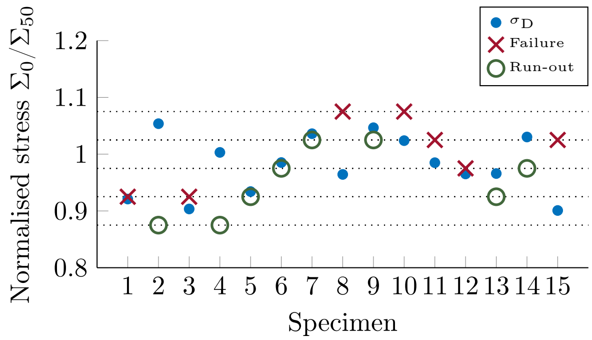
$$C_p = \frac{p}{\Sigma_{50}}, \tag{6}$$

with p the step size used during the staircase procedure and Σ_{50} the median fatigue limit of the initial distribution.

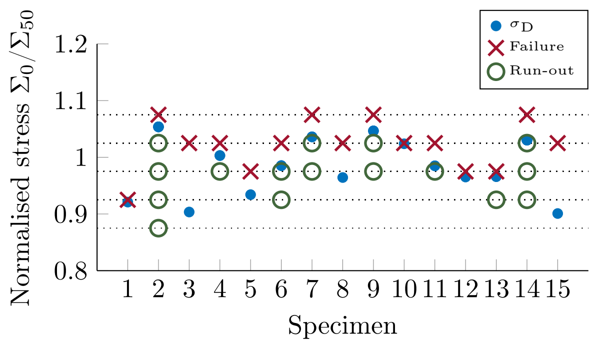
The influence of this step size has already been studied for the classic staircase procedure. Dixon and Mood [1] proposed an interval which was also highlighted with simulated staircases by Pollak [5]. Indeed, when the step size decreases, the number of specimens has to be increased to correctly oscillate around the median endurance limit. The method is then inefficient to determine the fatigue properties. Conversely, when the step size increases, the number of different stress levels on the staircase decreases (the specimens will simultaneously fail and survive in some extreme cases), and so the determination of the fatigue properties will be impossible. The acceptable step sizes are linked to the fatigue scatter. The ratio between the step size and the standard deviation should be considered to be within a range of 0.5 to 2. This ratio closely corresponds to the C_p/C_v ratio in this paper. Indeed, the Weibull PDF is asymmetric, the mean and the median of the distribution are not necessarily equal. The C_p/C_v ratio will be taken as equal to 1 during the first part of the investigation. Obviously, the standard deviation of the material is usually unknown before the staircase procedure is conducted. In practice, the step size is chosen from previous knowledge on a similar material or from the literature.

Moreover, the initial stress level of all simulated staircases is $\Sigma_{50} - \frac{3}{2}p$. Some studies in the literature show the effect of this initial stress level on the resulting fatigue property determination [9, 10]. This choice was made to reduce the number of specimens used to approach the median fatigue limit (first specimens in a staircase procedure).

The result of the specimen in the staircase, *i.e.* failure or run-out, only depends on the comparison between the randomly selected fatigue strength of the specimen σ_D and the applied stress Σ_0 during the procedure.



(a) Classic procedure



(b) With reuse of the unbroken specimens

Fig. 3. Staircase procedures with C_v=0.05 and C_p/C_v=1

The tested stress level of a specimen depends on the result of the previous specimen. Indeed, the stress level is increased if it was a run-out (so a survival after N_{ref} cycles) or is decreased if a failure occurred. A staircase using the classic procedure and 15 specimens is presented as an example (Fig. 3a).

For the variant proposed in this paper, all unbroken specimens are tested up to their failure. Without cumulative damage consideration, the stress level is increased until the applied stress is higher than the specimen fatigue strength. An example is presented using the same sample of 15 specimens (Fig. 3b). The goal is to study the possible gain of this procedure on the uncertainties of the fatigue property determination. Indeed, during fatigue tests, a broken specimen brings more possible information than a simple run-out, by analysing the two interesting load steps, the last run-out and the failure load. The objective of this staircase procedure is therefore to break all the specimens, to extract the maximum possible information from each sample.

2.3. Estimation of the fatigue properties

The Maximum Likelihood Estimation (MLE) is used in this paper to analyse the fatigue test results. This method is commonly used to analyse staircases [8, 7, 10, 12, 17] and more generally fatigue tests [18, 19, 20]. Moreover, the Dixon and Mood analysis [1] was determined from the MLE for a normal distribution. The comparison of the different evaluation techniques performed by Müller [7] shows that the results of the Dixon and Mood determination are similar to the MLE, for a classic staircase procedure with only censored data and for different distribution shapes.

The goal of the MLE is to combine the observations of each specimen in order to estimate the initial distribution. A likelihood function is created to evaluate the parameters of this distribution. Consequently, the distribution shape should be assumed.

In the general case, when the fatigue strengths of the n specimens are known and no censoring is performed, the likelihood function L is defined as

$$L(\alpha, \beta) = \prod_{i=1}^{n} f(\sigma_{D_i}), \tag{7}$$

where σ_{D_i} is the fatigue strength of the specimen i, f the PDF of the Weibull model, depending on the two parameters α and β (Eq. 2). The evaluation of the two distribution parameters is performed by maximizing this function. So, the determined Weibull distribution is the one with maximum likelihood from which the sample of specimens was randomly selected. In this paper, this result is then compared to the initial distribution, used for the specimen generation.

When performing real tests, the fatigue strengths of the specimens are obviously unknown. They could be estimated with assumptions on the fatigue curve shape and from the load history of the specimen (damage accumulation consideration). However, during a classic staircase, only run-out and failure are observed, and no fatigue strength calculation is performed on the broken specimen. For each specimen, the result from the staircase procedure brings some information on its fatigue strength. The observed probability P_i of the specimen i replaces the PDF in the MLE and corresponds to censored data. P_i is calculated using the CDF according to the result of the specimen i in the staircase.

Indeed, if the specimen i does not fail at the stress level Σ_{0_i}, the only information we derive from the test is that $\Sigma_{0_i} \leq \sigma_{D_i}$. It corresponds to a right censoring and the observed probability is then described using the CDF as

$$P_i(x \geq \Sigma_{0_i}) = 1 - F(\Sigma_{0_i}). \tag{8}$$

Likewise, if the specimen i fails at the stress level Σ_{0_i}, the only information we derive from the test is $\Sigma_{0_i} \geq \sigma_{D_i}$ (left censoring) and the observed probability is given by

$$P_i(x \leq \Sigma_{0_i}) = F(\Sigma_{0_i}). \tag{9}$$

When reloading specimens, another censoring is possible. Indeed, more information could be derived from the reloaded specimen i: its fatigue strength σ_{D_i} is inevitably greater than the stress levels with run-out (survival for N_{ref} cycles), and if no damage accumulation is considered, lower than the stress level

Σ_{0_i} for which the failure occurs. So, an interval censoring can be performed [21, 22, 23] and the observed probability on the specimen i is thus

$$P_i(\Sigma_{0_i} - p \leq x \leq \Sigma_{0_i}) = F(\Sigma_{0_i}) - F(\Sigma_{0_i} - p). \tag{10}$$

With no censoring data, the MLE is used with the most possible information from the sample, meaning that the uncertainties only come from the number of specimens in the sample (if we only include the uncertainties from the determination method, and not the uncertainties of the fatigue strength calculations). Moreover, with the MLE, the censoring data are clearly taken into account. The first question now concerns the minimum number of specimens required to determine the fatigue distribution parameters, with acceptable uncertainties. Indeed, in statistics, estimating the fatigue scatter with a small sample size, which is often the case when analysing real fatigue tests, is known to be an ambitious task and could lead to really large uncertainties.

3. Influence of the number of specimens per staircase on the fatigue property determination

In this section, the MLE is used to analyse the fatigue tests under multiple considerations. In order to obtain a statistical analysis, 1000 samples are generated, and staircases are simulated. For each sample, a median fatigue limit Σ_{50} and a coefficient of variation C_v are determined with the MLE. Consequently, distributions of these two parameters are obtained from the 1000 samples. The median response of the evaluation technique is obviously analysed with the medians of these distributions. The scatter in results, representing the uncertainties of the method, is evaluated with the 5^{th} percentile and the 95^{th} percentile of the distributions of the two fatigue parameters.

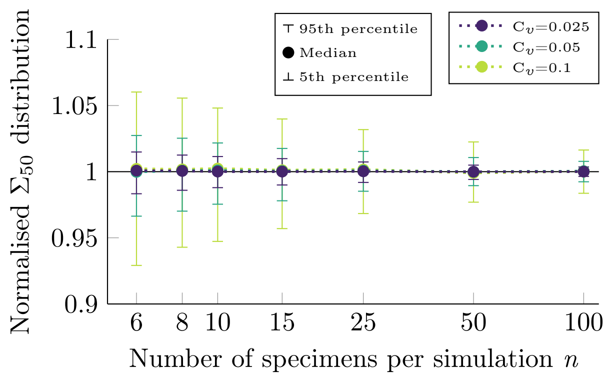
First, the ideal case is investigated, where all the fatigue strengths of the specimens are perfectly known. No censored data are considered. It represents the best case for which the maximum amount of information is extracted from the sample of n specimens. Consequently, it corresponds to the lowest achievable uncertainties for these n specimens. Then the classic staircase procedure is evaluated using only censored data (which mainly correspond to the Dixon and Mood analysis) and will represent the reference results. Next, the staircase procedure with reloading is also evaluated using only censored data. Finally, the different procedures are compared and the interest in reusing unbroken specimens is highlighted.

3.1. Ideal case: sample entirely known

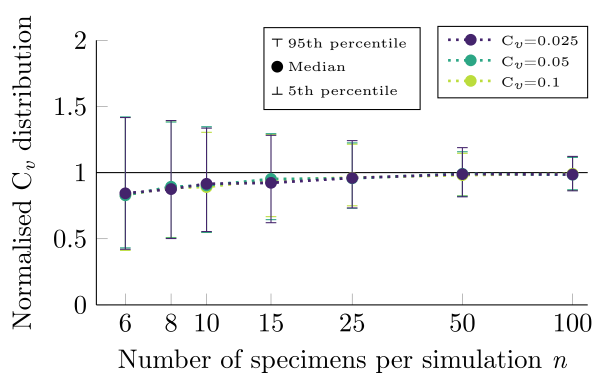
This is the case without censored data, so where the maximum information is extracted from the sample. For each condition of initial distribution and number of specimens, 1000 samples are simulated and are directly analysed using the MLE, without censored data. The results with three different conditions of fatigue property dispersion ($C_v = 0.025$, $C_v = 0.05$ and $C_v = 0.1$) are presented (Fig. 4) and normalised using the parameters of their initial distributions.

The uncertainties on the median fatigue limit (Fig. 4a), analysed with the interval between the 5^{th} and the 95^{th} percentiles, depend on the fatigue scatter of the initial distribution. The higher the coefficient of variation, the greater the uncertainties for the same number of specimens. However, the median response is always equal to the expected Σ_{50}. Logically, for the same initial distribution, when the number of specimens increases, the uncertainties decrease.

Concerning the distributions of C_v (Fig. 4b), the median response tends to be lower than the expected coefficient of variation for small size samples. Indeed, with a small number of specimens, the distribution tails are not correctly represented during the random selection. The determined coefficient of variation tends to be underestimated. The uncertainties can be united by normalising using the coefficient of variation of the initial distribution. Consequently, they only depend on the number of specimens n. Without surprise, the scatters



(a) Resulting distributions of the fatigue limit Σ_{50}



(b) Resulting distributions of the coefficient of variation C_v

Fig. 4. Distributions of the fatigue parameters determined using 1000 randomly selected samples, depending on the number of specimens per simulation - Ideal case with three different C_v for the initial distribution

in results are quite large for a small number of specimens, even for this ideal case. This is not a new result, but at least the lower achievable uncertainties are quantified for the two fatigue parameters as a function of the number of specimens n in the sample.

3.2. Classic staircase procedure

In this section, staircases are generated using the classic procedure and a load step of C_p/C_v=1. Only censored data are used for the fatigue determination. From the simulated staircases (Fig. 3a), there are two possible results for the specimens: run-out, so a right censoring (Eq. 8), or failure, and so a left censoring (Eq.9). This analysis closely corresponds to the Dixon and Mood method, for which the equations were determined using a normal distribution for the MLE [1, 7].

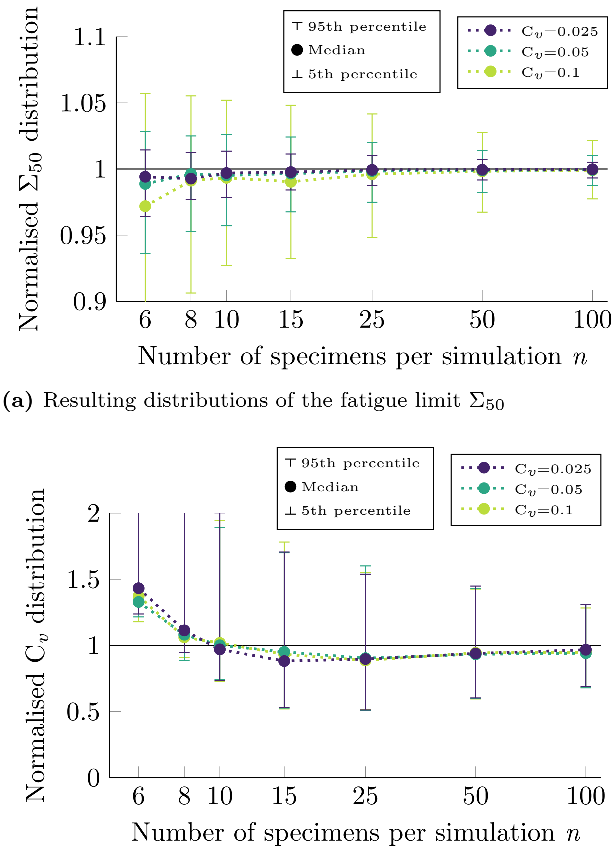
The results with three initial distributions are presented (Fig. 5). The Σ_{50} distributions are similar to the ideal case. Indeed, the median response tends to be the expected value and the uncertainties depend on the coefficient of variation of the initial distribution. Moreover, the uncertainties decrease when the number of specimens increases. Logically, the scatters in results seem to be greater than the ideal case. Indeed with censored data, less information is obtained from the sample and the uncertainties are therefore more numerous (for the same number of specimens).

When fewer specimens are used, the determination of the coefficient of variation is often impossible. Indeed, the MLE has insufficient information to evaluate the fatigue scatter. The calculated value is a bound of the numerical resolution, which is obviously not taken into account. Table 1 shows the proportion of C_v calculation for the three initial distributions as a function of the number of specimens n. The proportions of possible calculation are similar for the three initial distributions. The distributions of C_v are represented with only the relevant obtained values. So, they do not have the same amount of data depending on the number of specimens. The distributions (Fig. 5b) should be analysed with this consideration and thus with the knowledge of the proportion of C_v calculation (Table 1). The results can also be united by normalising using the coefficient of variation of the initial distribution. For samples with 10 specimens or less, the proportion of calculation is less than 50% and the median response of the evaluation technique tends to be overestimated. The scatter in results is very large, even with large samples (at least 50 specimens recommended by Dixon and Mood for the fatigue scatter determination).

This procedure and evaluation method are really robust to determine the median fatigue limit using a few specimens. However, the fatigue scatter is hardly evaluated. This is not a new result, but this study allows the quantification of the uncertainties and sets the reference result for the comparison with the staircase procedure using reloaded specimens.

3.3. Staircase procedure with reloading

In this section, the staircases are also simulated with a load step of C_p/C_v=1. There are two possible results for the specimens (Fig. 3b): failure at their first



(a) Resulting distributions of the fatigue limit Σ_{50}

(b) Resulting distributions of the coefficient of variation C_v

Fig. 5. Distributions of the fatigue parameters determined using 1000 simulated staircases, depending on the number of specimens per simulation - Classic procedure with three different C_v for the initial distribution and C_p/C_v=1

loading level, so left censoring (Eq. 9), or failure after several loading levels, so an interval censoring (Eq. 10).

The results with three initial distributions are presented (Fig. 6). The proportion of C_v calculation is also shown (Table 2). The same conclusions as previously can be drawn on the median fatigue limit Σ_{50}. However, the uncertainties seem to be lower than for the classic procedure, but still more numerous than the ideal case. Indeed, the amount of information derived from the samples is higher than when using the classic procedure for the same number of specimens.

Likewise, the same conclusions as previously can be formulated for the coefficient of variation C_v. The proportion of possible C_v calculation is increased as compared to the classic procedure (Table 1 and Table 2) because more information is obtained using the interval censoring method than with a simple

Table 1
Proportion of possible C_v determination for the 1000 simulated staircases using the classic procedure depending on the number of specimens per simulation

n	**Proportion of C_v calculation**		
	C_v=0.025	C_v=0.05	C_v=0.1
6	0.24	0.24	0.23
8	0.40	0.35	0.36
10	0.49	0.51	0.52
15	0.71	0.73	0.75
25	0.90	0.91	0.91
50	0.99	0.99	0.99
100	1	1	1

right censoring for unbroken specimens. The uncertainties of the coefficient of variation are still numerous but are significantly reduced in comparison with the classic procedure (Fig. 5b and Fig. 6b).

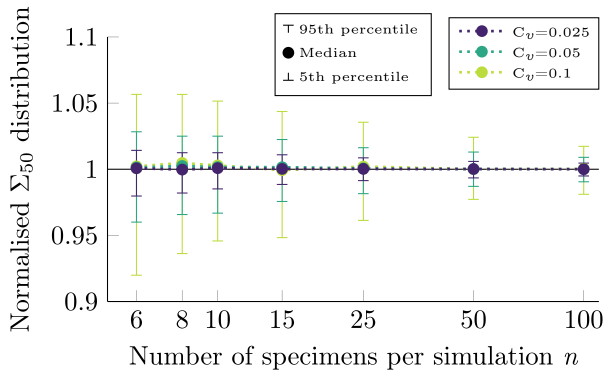
Table 2
Proportion of possible C_v determination for the 1000 simulated staircases with reloading of unbroken specimens and the censoring interval depending on the number of specimens per simulation

n	**Proportion of C_v calculation**		
	C_v=0.025	C_v=0.05	C_v=0.1
6	0.70	0.70	0.71
8	0.84	0.82	0.85
10	0.87	0.91	0.92
15	0.97	0.97	0.98
25	1	1	1
50	1	1	1
100	1	1	1

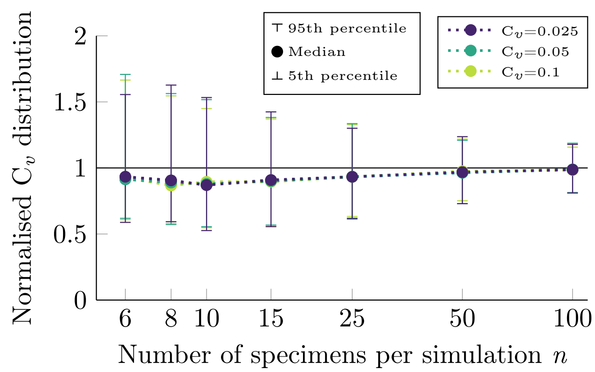
3.4. Interest in reusing unbroken specimens

The interest in reusing unbroken specimens could already be highlighted in the previous part. To help the comparison along, the results of the ideal case and of the two staircase procedures are plotted on the same graph for one initial distribution (defined with C_v=0.05) and using the step size C_p/C_v=1 (Fig. 7).

The results for the normalised median fatigue limit Σ_{50} (Fig. 7a) show that the uncertainties are reduced when the reloading is performed. This is linked again to the amount of data extracted from the sample. The gain on the uncertainties of the median fatigue limit is not so high and is mainly observed for small sample sizes. This is still linked to the initial distribution: with a more dispersive distribution, the observed gain would be much higher.



(a) Resulting distributions of the fatigue limit Σ_{50}



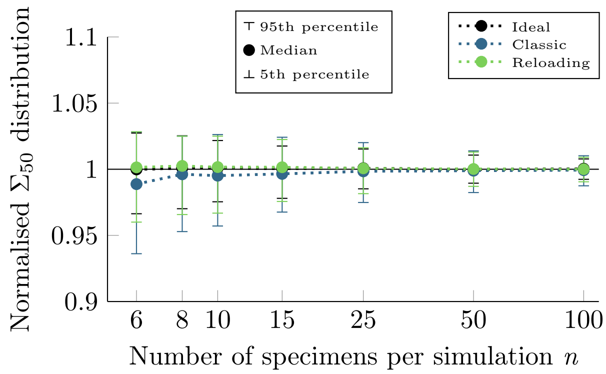
(b) Resulting distributions of the coefficient of variation C_v

Fig. 6. Distributions of the fatigue parameters determined using 1000 simulated staircases, depending on the number of specimens per simulation - Staircase with reloading and the interval censoring method with three different C_v for the initial distribution and C_p/C_v=1

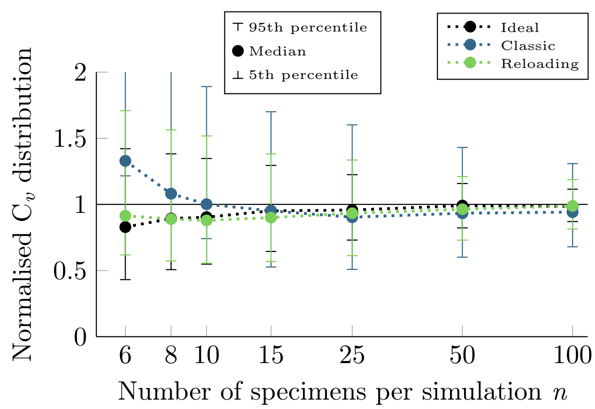
Concerning the normalised coefficient of variation C_v (Fig. 7b), the results are not dependent on the coefficient of variation of the initial distribution. The results have to be analysed with the knowledge of the proportion of C_v calculation (Table 3). The interest in reusing unbroken specimens is highlighted - meaning that with a small sample size the C_v calculation is more often possible, and the uncertainties are significantly reduced for the same number of specimens used. The uncertainties become closer to the ideal case.

However, an assumption on the damage accumulation should be considered when reloading unbroken specimens. Indeed, the interval censoring does not take into account the cumulative damage and a validity domain should be defined to use this method with damage consideration (Section 5).

Another advantage of the reloading procedure is that all the specimens are broken. If relevant assumptions can be stated on the shape of the S-N curve, and possibly on the cumulative damage when unbroken specimens are reloaded,



(a) Resulting distributions of the fatigue limit Σ_{50}



(b) Resulting distributions of the coefficient of variation C_v

Fig. 7. Distributions of the fatigue parameters determined using 1000 simulated staircases, depending on the number of specimens per simulation - Comparison of the procedures and determination methods with C_v=0.05 and C_p/C_v=1

the number of cycles at failure can be used to evaluate the allowable stress of the specimens. More information is extracted from each specimen and if this allowable stress is assimilated to the fatigue strength of the specimen, the MLE could be used without data censoring (Eq. 7). The uncertainties are thus minimised for the estimation method (without censoring), but others appear for the calculation of this allowable stress. If the assumptions are correct, the fatigue strengths are perfectly known, which corresponds to the ideal case. The effect of an incorrect fatigue strength determination is also studied in this paper using a certain cumulative damage law (Section 6). A sensitivity study is then performed on the parameter describing this fatigue strength determination.

But first, a cumulative damage law has to be defined and staircases with damage consideration must be simulated.

Table 3
Comparison of the proportion of possible C_v determination for the 1000 simulated staircases with the different determination methods depending on the number of specimens per simulation

n	**Proportion of C_v calculation**		
	Classic	Reloading	Ideal
6	0.24	0.70	0.97
8	0.35	0.82	0.99
10	0.51	0.91	1
15	0.73	0.97	1
25	0.91	1	1
50	0.99	1	1
100	1	1	1

4. Damage consideration in the staircase simulation

During the simulation of reloaded specimens with damage consideration, the applied stress is no longer directly compared to the randomly selected fatigue strength. A number of cycles at failure N_{r_i} is calculated for each specimen, as a function of its fatigue strength and its previous loading. This number is then compared to the reference number of cycles N_{ref} to determine the result of the specimen in the staircase procedure at a certain stress level (run-out or failure).

In order to perform the calculation, a cumulative damage law should be assumed. The shape of the S-N curve should be stated to determine the link between the applied stress and the number of cycles at failure.

In this paper, the Basquin model [24] has been chosen to describe the fatigue curve around the reference number of cycles N_{ref}. This is a simple model as it is a straight line in a logS-logN plan (Fig. 1) and every S-N curve could be locally approximated using this model. It is defined as

$$N_{r_i} {\Sigma_{0_i}}^{b_m} = A, \tag{11}$$

with N_{r_i} the number of cycles at failure for the stress level Σ_{0_i}, and where A and b_m are two material parameters.

During the simulation, the number of cycles at failure for the specimen i at its first applied stress level could be determined using

$$N_{r_i} = N_{\text{ref}} \left(\frac{\sigma_{D_i}}{\Sigma_{0_i}} \right)^{b_m}. \tag{12}$$

Only the parameter b_m is required and it simply represents the slope of the S-N curve around N_{ref}. Obviously in this case, the comparison between the calculated number of cycles N_{r_i} and the reference number of cycles N_{ref} is identical to the comparison between the randomly selected fatigue strength σ_{D_i} and the applied stress Σ_{0_i}.

If a run-out occurs, the unbroken specimen is then reloaded on the next stress level. To evaluate the damage accumulation, the basic linear rule of Miner [25] has been chosen and using the Basquin model (Eq. 11) for the S-N curve, the damage is calculated using

$$D = \sum \frac{n_i}{N_i} = \sum n_i \frac{{\Sigma_{0_i}}^{b_m}}{A}. \tag{13}$$

In practice and for this paper, an assumption is formulated: the damage accumulation only occurs for the last two stress levels. Thus, the number of cycles at failure for the stress level Σ_{0_i} is obtained using

$$N_{r_i} = N_{\mathrm{ref}}\ \frac{{\sigma_{D_i}}^{b_m} - (\Sigma_{0_i} - p)^{b_m}}{{\Sigma_{0_i}}^{b_m}}. \tag{14}$$

The slope b_m of the S-N curve and the step size p are the two parameters impacting the damage accumulation. In order to make the analysis easier, a linear coefficient C_{bm} is created from the slope of the S-N curve. It represents the decrease in the median fatigue strength on one decade of cycles and is calculated using

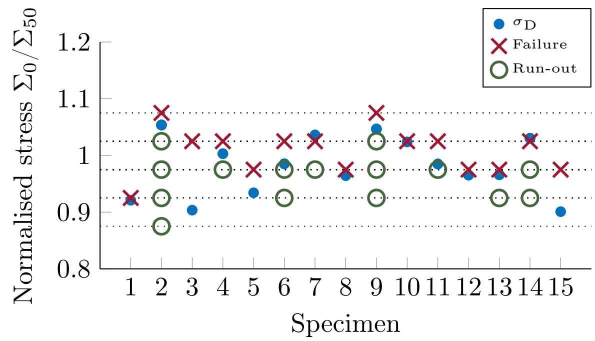
$$\mathrm{C}_{bm} = 10^{1/b_m} - 1. \tag{15}$$

Two staircases with damage consideration are presented (Fig. 8). They are generated from the same sample as previously (Fig. 3) and the same parameters: C_v=0.05 and $\mathrm{C}_p/\mathrm{C}_v$=1.

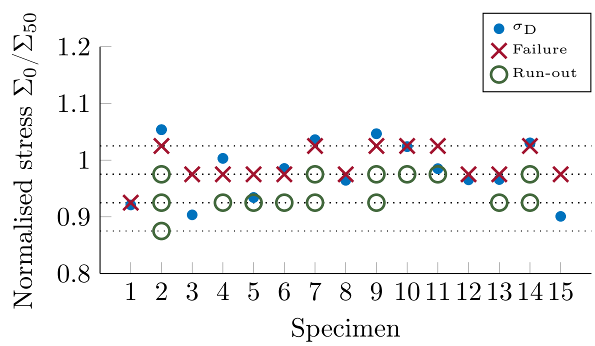
The damage accumulation logically impacts the staircase proceedings. For example, with C_{bm}=0.1, the failure of Specimen 7 occurs for a stress level under its randomly selected fatigue strength (Fig. 8a). This will have an effect on the stress level on which the censored data occur. Moreover,the simulated number of cycles will be different and thus have an effect on the fatigue strength calculation. In a more extreme case with C_{bm}=0.2, the simulated staircase is completely different. The use of the interval censoring method is then questioned, and a validity domain should be defined, *i.e.* the values of C_v, C_p and C_{bm} for which the cumulative damage does not affect the staircase proceedings, and thus for which the interval censoring can still be used (Section 5). In the extreme case with C_{bm}=0.2, the calculation of all fatigue strengths seems to be preferable. However, many more assumptions are required in order to achieve that. A parametric study is presented in Section 6 on the effect of an incorrect fatigue strength determination.

5. Validity of the interval censoring method with damage consideration for the staircase simulations

The objective of this section is to quantify the bias induced with the cumulative damage on the interval censoring method. The goal is therefore to evaluate the set of parameters (C_v, C_p and C_{bm}) for which the cumulative damage has little effect on the staircase proceedings.



(a) Simulation parameter C_{bm}=0.1



(b) Simulation parameter C_{bm}=0.2

Fig. 8. Staircase simulations with damage consideration for reloaded specimens - Influence of the simulation parameter C_{bm}, with C_v=0.05 and C_p/C_v=1

First, this domain of validity is investigated for a large number of specimens ($n = 100$) assuming that if the predictions are incorrect, there is no reason that they would be correct for a smaller sample size. The effect of the step size is studied and a relation between C_p and the coefficient of variation C_v of the initial distribution is proposed. Then, the effect of the damage accumulation (through the parameter C_{bm}) is evaluated. Finally, the influence of the number of specimens is investigated in order to validate the defined domain.

5.1. Effect of step size

For the classic procedure, Dixon and Mood [1] proposed the range

$$2 \geq \frac{C_p}{C_v} \geq 0.5. \tag{16}$$

This result was also found by Pollak [5] with simulations for different numbers of specimens per staircase.

As previously, the results of the 1000 simulations are presented with the median and the scatter in results with the 5$^{\text{th}}$ and the 95$^{\text{th}}$ percentiles. Simulations are performed for two initial distributions (C_v=0.05 and C_v=0.1) with a sample size of $n = 100$ specimens. The effect of the step size is evaluated by performing simulations with different C_p/C_v ratios. The results for the normalised median fatigue limit (Fig. 9) and for the normalised coefficient of variation (Fig. 10) are presented.

For smaller step sizes, the median response of Σ_{50} is reasonably affected by the step size due to the fact that the damage calculation is only performed on the last two levels. However, the median response of C_v is strongly affected with the diminution of the step size and the increase in C_{bm}. The lower boundary of $C_p/C_v = 0.5$ proposed by Dixon and Mood for the classic procedure is linked to a minimal number of specimens in order to correctly oscillate around the median fatigue limit. Indeed, with a small step size, more specimens are required to increase or decrease the loading step Σ_0 rapidly in the staircase procedure [5]. In the case of staircases with reloaded specimens, a similar boundary can be set due to the damage consideration.

Above $C_p/C_v = 2$, the scatter in results is affected by the step size. It is due to the fact that even for a large number of specimens, all of them could be on the same two or three stress levels. In order to reach a good accuracy on the staircase analysis, the step size should not be too big so as to have enough different stress levels and allow the determination of the properties.

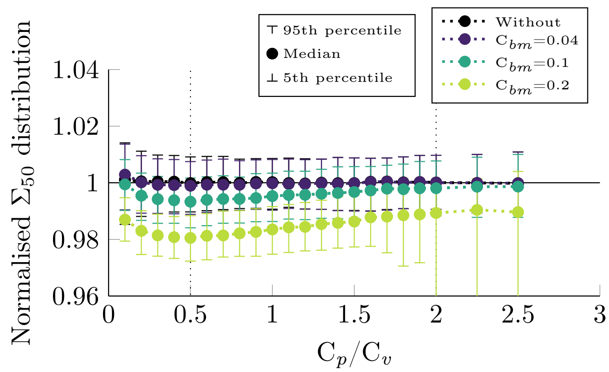
In conclusion, the same range (Eq. 16) as the classic procedure could be defined for the choice of the step size. As mentionned previously, the upper boundary ($2 \geq C_p/C_v$) ensures that there are enough stress levels, thus different intervals, to allow the determination of the fatigue properties. On the other hand, the lower bound ($C_p/C_v \geq 0.5$) is set to ensure that the cumulative damage has little effect on the determination of the parameters. It still depends on the C_{bm} parameter and another condition has to be set. Indeed, the case with C_v=0.05 seems more affected by the increase in the C_{bm} parameter than the case with C_v=0.1 (Fig. 10).

5.2. Impact of the damage accumulation

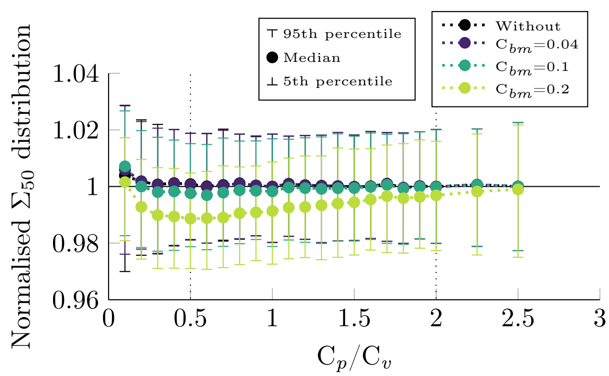
The damage accumulation mostly affects the determination of the coefficient of variation. Indeed, it tends to gather all the specimens on the same stress levels (Fig. 8b), and so the coefficient of variation is underestimated. As the scatter in the results for C_v seems to be quite constant (in the correct range of C_p/C_v), only the median response is studied.

Median values of C_v distribution are determined for three different initial distributions and for a large range of C_p and C_{bm} in order to evaluate the impact of the damage accumulation. The results are represented as a function of C_p/C_{bm} (Fig. 11).The trends are clear and all the results for the different initial distributions are united. Therefore, to avoid the cumulative damage having an effect on the staircase proceedings, the step size should also be chosen in accordance to the slope of the S-N curve such that

$$\frac{C_p}{C_{bm}} \geq 0.5. \tag{17}$$



(a) Resulting distributions of the fatigue limit Σ_{50} for an initial distribution with C_v=0.05



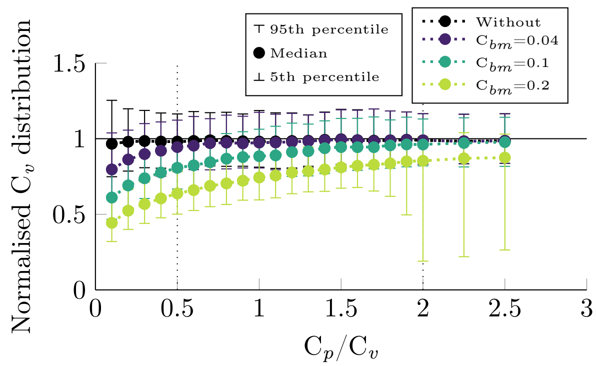
(b) Resulting distributions of the fatigue limit Σ_{50} for an initial distribution with C_v=0.1

Fig. 9. Effect of the step size (with the C_p/C_v ratio) on the fatigue limit distributions determined using 1000 simulated staircases of $n = 100$ specimens and the interval censoring method - Comparison with and without damage consideration

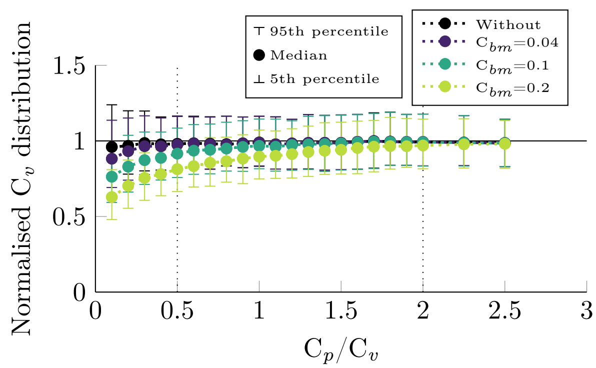
The domain of validity of the interval censoring is thus defined for the chosen cumulative law (Eq. 16 and Eq. 17). It means that for some conditions of C_{bm} and C_v there is no possible step size to perform a staircase procedure. It corresponds to really extreme configurations with a low dispersion or a major slope in the S-N curve ($C_{bm} \geq 4\ C_v$), which is generally not the case in the HCF domain.

5.3. Influence of the number of specimens per staircase

In order to confirm this validity domain, simulations are performed for an initial distribution of C_v=0.05 and for C_p/C_v=1. Four conditions are considered: without damage accumulation and with three different C_{bm}. The goal is to evaluate the validity domain for smaller numbers of specimens. The effect of the C_p/C_{bm} ratio on the fatigue property determination is represented as a



(a) Resulting distributions of the coefficient of variation C_v for an initial distribution with C_v=0.05



(b) Resulting distributions of the coefficient of variation C_v for an initial distribution with C_v=0.1

Fig. 10. Effect of the step size (with the C_p/C_v ratio) on the coefficient of variation distributions determined using 1000 simulated staircases of $n = 100$ specimens and the interval censoring method - Comparison with and without damage consideration

function of the number of specimens (Fig. 12). The results for the coefficient of variation (Fig. 12b) have to be interpreted with the proportion of C_v calculation as a function of the number of specimens used during the staircases (Table 4).

As expected, when the condition stated on the choice of the step size in accordance to the S-N slope (Eq. 17) is respected, the staircase proceedings are not affected by the cumulative damage. So, the resulting distributions of the two parameters are not affected and are close to the case without damage consideration. When the condition is not respected, the cumulative damage gathers the specimens on fewer levels for lower stress amplitudes. The median fatigue limit tends to be underestimated. Moreover, the determination of the fatigue scatter is less often possible, and the coefficient of variation also tends to be underestimated.

The interval censoring method is interesting because no assumption is needed

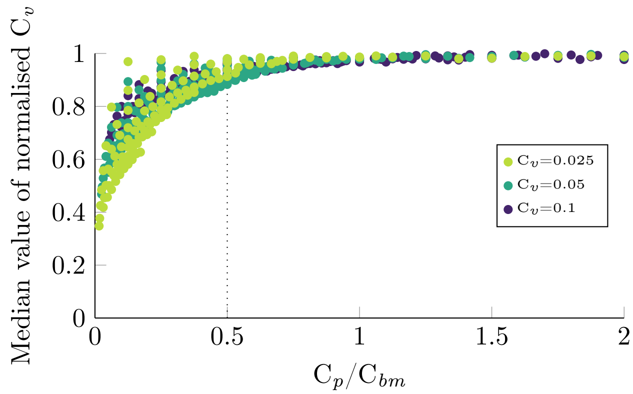


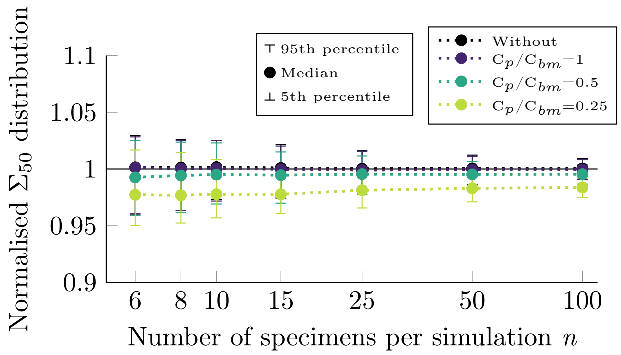
Fig. 11. Effect of the C_p/C_{bm} ratio on the median value of the coefficient of variation distributions determined using 1000 simulated staircases of $n = 100$ specimens and the interval censoring method - Comparison for three initial distributions using various step sizes C_p and simulation parameters C_{bm}

Table 4
Effect of the C_p/C_{bm} ratio on the proportion of possible C_v determination for the 1000 simulated staircases depending on the number of specimens per simulation

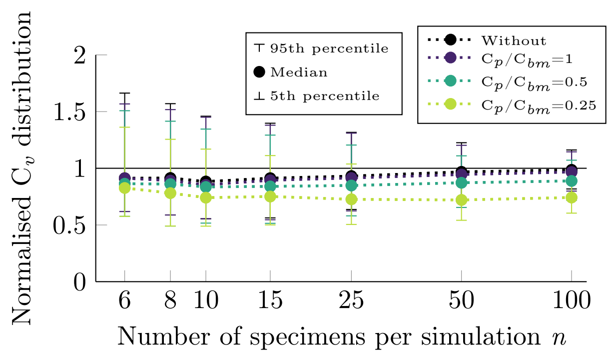
n	Proportion of C_v calculation			
	Without	$\frac{C_p}{C_{bm}}$=1	$\frac{C_p}{C_{bm}}$=0.5	$\frac{C_p}{C_{bm}}$=0.25
6	0.70	0.70	0.65	0.42
8	0.82	0.83	0.79	0.56
10	0.91	0.89	0.86	0.63
15	0.97	0.98	0.96	0.78
25	1	1	1	0.92
50	1	1	1	0.99
100	1	1	1	1

during the fatigue property determination. However, the step size of the procedure must be wisely chosen in order to obtain exploitable staircases, with enough different stress levels, and by minimizing the effect of the cumulative damage (Eq. 16 and Eq. 17). The coefficient of variation and the slope of the S-N curve around N_{ref} should be estimated, with prior knowledge or experience of similar materials.

After performing staircases, it is still possible to evaluate the fatigue strengths of the specimens, in order to use the estimation method without censoring data. Important assumptions must be formulated, especially on the cumulative damage rule. However, the uncertainties due to the estimation method are minimized (no censoring data). A study is presented on the use of the fatigue strength calculation method, with different S-N curve slopes in order to evaluate the sensitivity of this parameter.



(a) Resulting distributions of the fatigue limit Σ_{50}



(b) Resulting distributions of the coefficient of variation C_v

Fig. 12. Effect of the C_p/C_{bm} ratio on the distributions of the two fatigue parameters determined using 1000 simulated staircases, depending on the number of specimens per simulation - Interval censoring method with C_v=0.05, C_p/C_v=1 and different simulation parameters C_{bm}

6. Determination of fatigue strengths with damage consideration

The goal of this section is to quantify the bias induced with an incorrect estimation of the fatigue strengths of the specimens. The same damage cumulative rule is used for the simulation of the staircases and for the fatigue property determination. In order to calculate the allowable stress, some assumptions are required. First, the S-N curve shape has to be stated, and the cumulative damage rule has to be chosen. Some fatigue analyses use similar assumptions in order to evaluate the fatigue properties, for example in a Locati procedure [20].

The Basquin model (Eq. 11) was chosen to describe the S-N curve. Indeed, the slope of the S-N curve can be quite easily estimated using previous knowledge or experience on comparable material. To calculate the allowable stress of the reloaded specimens, the Miner linear rule (Eq. 13) is complemented with the Basquin model. Thus, the allowable stress of each specimen is considered equal to its fatigue strength.

Staircase proceedings are generated and for each specimen i, a number of cycles to failure N_{r_i} is calculated from its random selected fatigue strength and using the b_m parameter (Eq. 12 or Eq. 14).

Using a b_d slope in order to approximate the S-N curve, an allowable stress is calculated for each specimen. If the failure occurs during the first stress level Σ_{0_i}, the allowable stress is given by

$$\sigma_{N_{\mathrm{ref}_i}} = \Sigma_{0_i} \left(\frac{N_{r_i}}{N_{\mathrm{ref}}} \right)^{1/b_d}. \tag{18}$$

Likewise, if the failure occurs after reloading, the allowable stress is evaluated using

$$\sigma_{N_{\mathrm{ref}_i}} = \left((\Sigma_{0_i} - p)^{b_d} + \frac{N_{r_i}}{N_{\mathrm{ref}}} {\Sigma_{0_i}}^{b_d} \right)^{1/b_d}. \tag{19}$$

If the parameters are perfectly known, meaning that $b_d = b_m$, the allowable stress is exactly the randomly selected fatigue strength of the specimen. It is therefore the ideal case (see Section 3.1).

In order to perform the analysis more easily, a similar coefficient is created to represent the decrease in the median fatigue strength on one decade of cycles and is calculated using

$$\mathrm{C}_{bd} = 10^{1/b_d} - 1. \tag{20}$$

The goal of this section is to determine the sensitivity of the parameter representing the S-N curve slope of the fatigue determination method. Staircases are generated with various C_{bm} parameters and the fatigue property determination is performed with different C_{bd} values. For each condition of C_{bm} and C_{bd}, 1000 simulations are done in order to evaluate the median response of the determination of the two fatigue parameters.

Firstly, the sensitivity of the S-N curve slope for the fatigue strength calculation is presented for a large number of specimens per sample ($n = 100$) for different initial distributions and step sizes. Then, the same analysis is performed on the determination of the coefficient of variation. Finally, this sensitivity is evaluated for a smaller number of specimens.

6.1. Sensitivity of the fatigue strength calculation parameter for the fatigue limit determination

The different simulations allow the representation of mappings, showing the evolution of the median response of the Σ_{50} determination (obtained from 1000 simulations of 100 specimens) as a function of the C_{bm} and C_{bd} parameters. The simulations were performed for two initial fatigue distributions, C_v=0.05 (Fig. 13) and C_v=0.1 (Fig. 14). For each of them, three different step sizes were evaluated. The cases where the normalised fatigue limit equals 1 are highlighted and match generally the condition C_{bd}=C_{bm}. The cases where the normalised Σ_{50} are 0.99 and 1.01 are also represented in order to facilitate the sensitivity analysis.

When the step size increases, the sensitivity of the parameter is higher (the surface inside the interval [0.99 1.01] tends to decrease). This is due to the fact that with a bigger step size and with the chosen cumulative rule, the simulated numbers of cycles to failure tend to be smaller (the applied stress tends to be further than the randomly selected fatigue strength). The determination method is much more affected by an incorrect estimation of the S-N curve slope.

Moreover, the sensitivity increases with the fatigue scatter of the initial distribution. Indeed, it is a similar effect as with the step size. If the initial distribution is more dispersive, the applied stress level during the stress procedure is possibly further from the fatigue strength of the specimen and the simulated number of cycles is then lower. Consequently, an incorrect estimation of the S-N curve slope will have a bigger impact on the median response of the Σ_{50} determination method for more dispersive distribution.

This method is robust in the median fatigue limit determination, the sensitivity is quite low. However, this is a simulation-based investigation with many assumptions and the results are idealised. Indeed, the damage rule law is perfectly known and is assumed to be the same for the simulation and the fatigue strength determination. In order to reduce the sensitivity, it would not be wise to reduce the step size because it would increase the damage accumulation. Besides, the acceptable range of the C_{bd} parameter to result in an error of less than 1% for the median response of the Σ_{50} determination is still large with bigger step sizes.

6.2. *Sensitivity of the fatigue strength calculation parameter for the fatigue scatter determination*

The different simulations allow the representation of similar mappings for the coefficient of variation C_v. The simulations were performed for different step sizes and for two initial distributions: C_v=0.05 (Fig. 15) and C_v=0.1 (Fig. 16). These mappings represent the median response of the determination method, obtained from 1000 staircase simulations of 100 specimens, as a function of the calculation parameter C_{bd} and the simulation parameter C_{bm}. The cases where the normalised coefficient of variation equals 1 are shown and match the condition C_{bd}=C_{bm}. The acceptable range for the median value of C_v is determined as being within a 10% margin of error of the target value. Thus, the cases where the normalised coefficients of variation are 0.9 and 1.1 are highlighted.

The median response of the C_v determination seems only to be affected by the step size. Indeed, for the same C_p/C_v ratio, the mappings are similar (for example Figure 15b and Figure 16b).

Unlike the fatigue limit Σ_{50}, when the step size increases, the sensitivity of the median response decreases. Indeed, it is due to the "projection" (Basquin law) used to estimate the allowable stress at the number of cycles N_{ref}. When the step size increases, there are generally fewer different stress levels, and the failures occur at lower numbers of cycles. Consequently, the fatigue scatter determination is less affected by this projection, although the fatigue limit determination tends to be more impacted. Conversely, when the step size decreases, the number of different stress levels increases. However, the simulated numbers of cycles at failure tend to gather around N_{ref} and thus, the sensitivity of the fatigue scatter determination is increased using this projection.

In any case, the effects of the step size on the sensitivity of the fatigue strength calculation method are similar. The acceptable range resulting in an error of less than 10% for the median response can be estimated (for the condition C_p/C_v=1.0):

$$1.35 \geq \frac{C_{bm}}{C_{bd}} \geq 0.77. \tag{21}$$

For instance, when the simulation is performed with C_{bm}= 0.15, the C_{bd} parameter should be in the range of 0.11 to 0.20. When the C_{bm} is lower, for example C_{bm}= 0.03 then the C_{bd} should be between 0.022 and 0.04. The sensitivity is thus greater for a lower S-N fatigue slope. However, in this case, the cumulative damage is reduced, and the interval censoring method is probably more appropriate, with a large C_p/C_{bm} ratio (see Section 5.2).

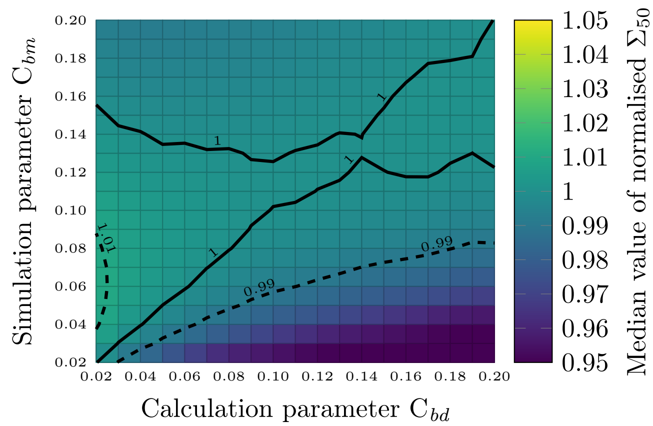
6.3. Influence of the number of specimens per staircase

In this section, the analysis is performed for smaller sample sizes in order to validate the previous observations on samples of $n = 100$ specimens. For each sample size, 1000 staircases are simulated with an initial distribution defined by C_v=0.05, a step size of C_p/C_v=1 and the cumulative damage parameter C_{bm}= 0.05. Then, the fatigue determination is performed using four values of C_{bd}. One of these values is C_{bd}= 0.05 and is thus the ideal case. C_{bd}= 0.03 and C_{bd}= 0.07 correspond to conditions close to the boundaries of the acceptable range (Eq. 21). The fourth value C_{bd}= 0.10 is outside the acceptable range. The distributions of the fatigue parameters are presented for the different C_{bd} values (Fig. 17) as a function of the number of specimens.

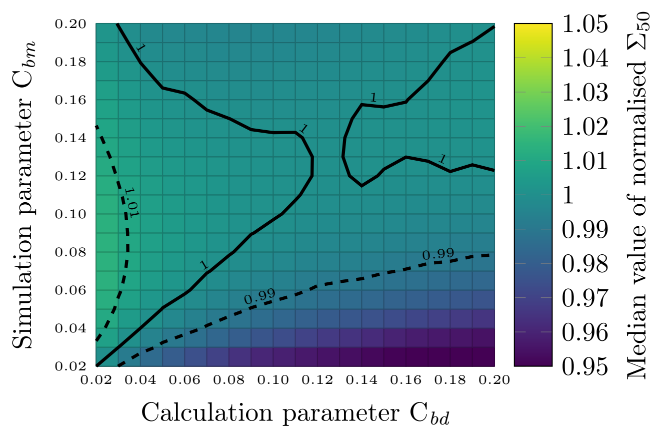
As previously stated, an incorrect calculation parameter has little impact on the Σ_{50} determination (Fig. 17a), even for a small number of specimens. The scatters in results (represented with the size of the interval between the 5^{th} and the 95^{th} percentiles) seem to be moderately affected. The median response, and so inevitably all the values, are just slightly different from the ideal case, represented by C_{bm}=C_{bd}.

Concerning the coefficient of variation (Fig. 17b), the same conclusions can be formulated, although the scatter in results seems more affected for small sample sizes. However, the same tendency is observed, and it seems that all the values are just shifted due to the bad approximation of the S-N curve slope.

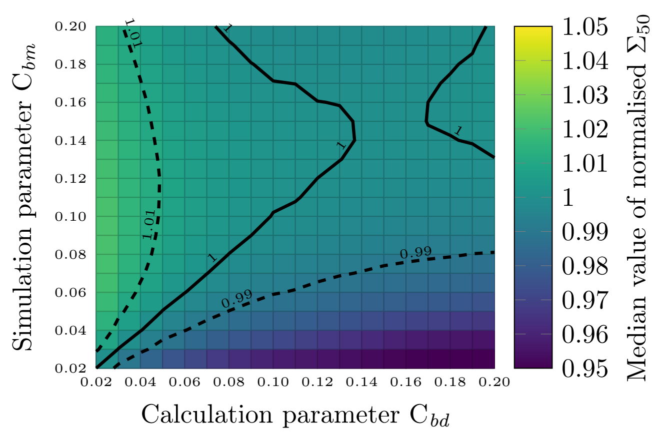
In order to complete the analysis, mappings of the sensitivity on the median response of the determination of the two fatigue parameters are obtained for a smaller sample size of $n = 15$ (Fig. 18). The simulations are performed with an initial distribution C_v=0.05 and a staircase procedure with C_p/C_v=1. The mapping for the median response on the Σ_{50} determination (Fig. 18a) can be compared to that obtained for the same conditions using a larger number of specimens per simulation (Fig. 13b). In the same way, the mapping representing the median response of the C_v determination (Fig. 18b) can be compared to that obtained using $n = 100$ specimens (Fig. 15b). The median responses



(a) C_p/C_v=0.5

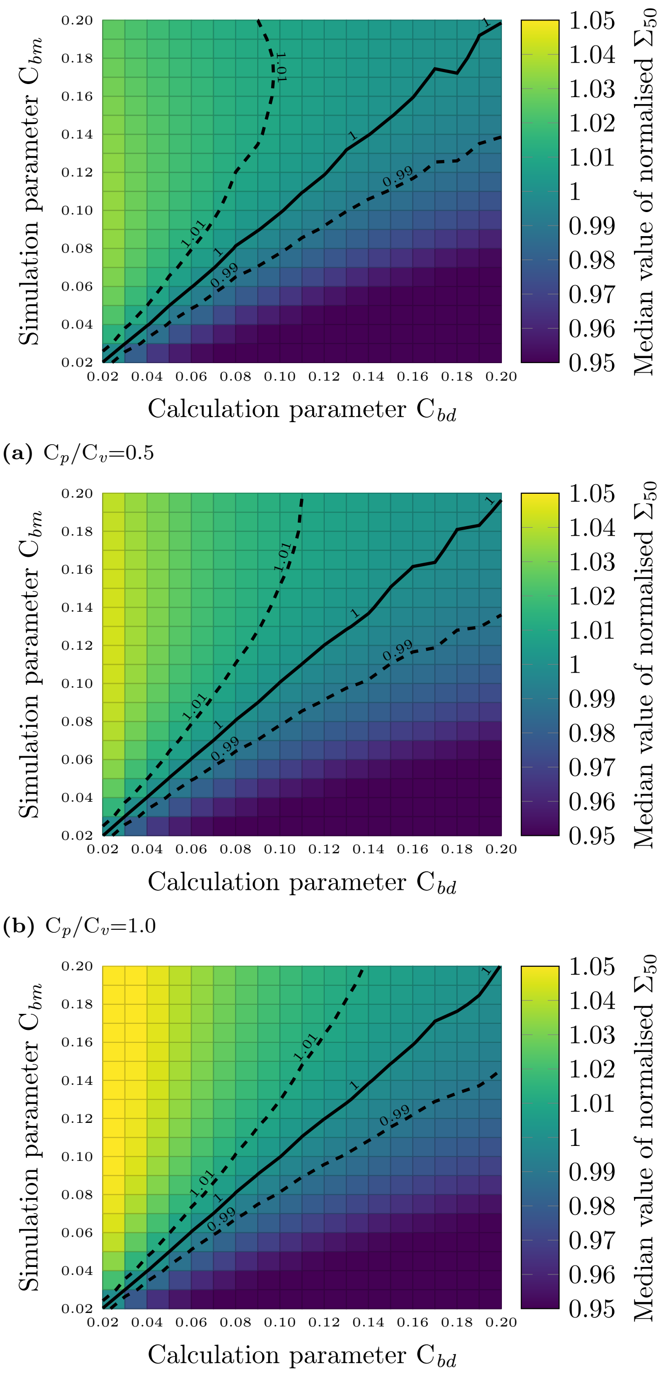


(b) C_p/C_v=1.0



(c) C_p/C_v=1.5

Fig. 13. Effect of the step size (with the C_p/C_v ratio) on the median value mappings of Σ_{50} distributions determined using 1000 simulated staircases of $n = 100$ specimens - Sensitivity of the C_{bd} parameter on the fatigue strength calculation method using an initial distribution with C_v=0.05

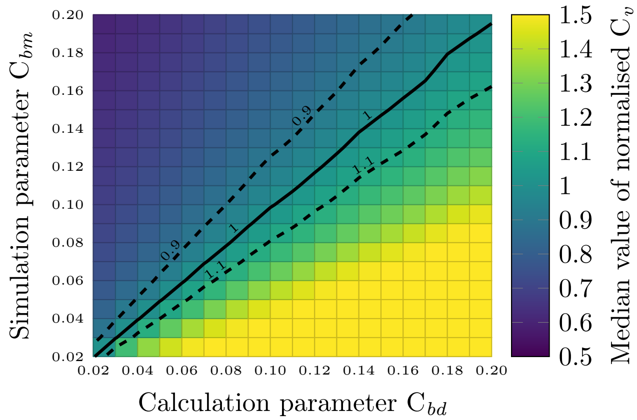


(a) C_p/C_v=0.5

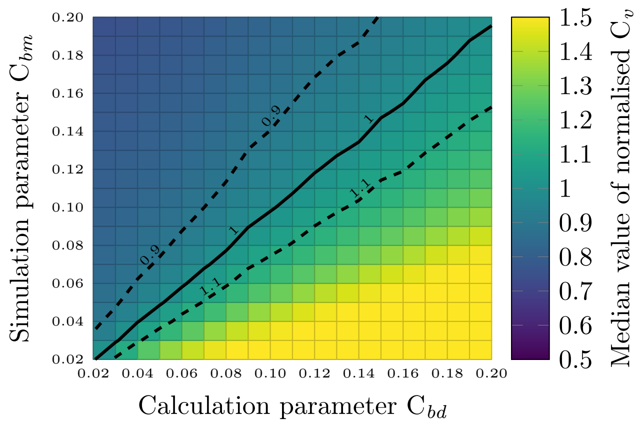
(b) C_p/C_v=1.0

(c) C_p/C_v=1.5

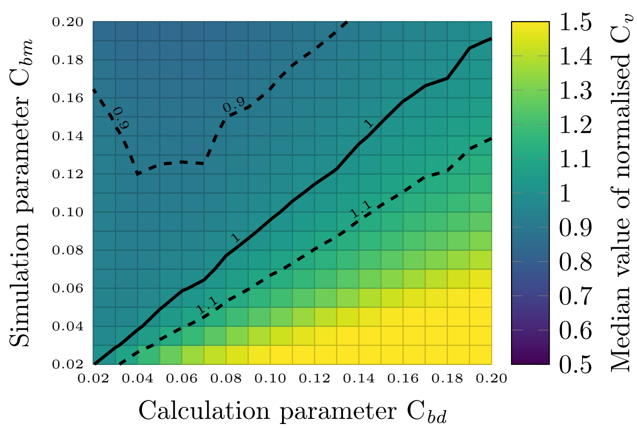
Fig. 14. Effect of the step size (with the C_p/C_v ratio) on the median value mappings of Σ_{50} distribution determined using 1000 simulated staircases of $n = 100$ specimens - Sensitivity of the C_{bd} parameter on the fatigue strength calculation method using an initial distribution with C_v=0.1



(a) C_p/C_v=0.5

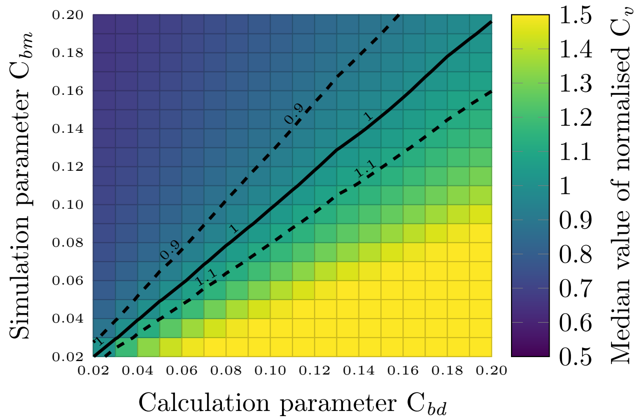


(b) C_p/C_v=1.0

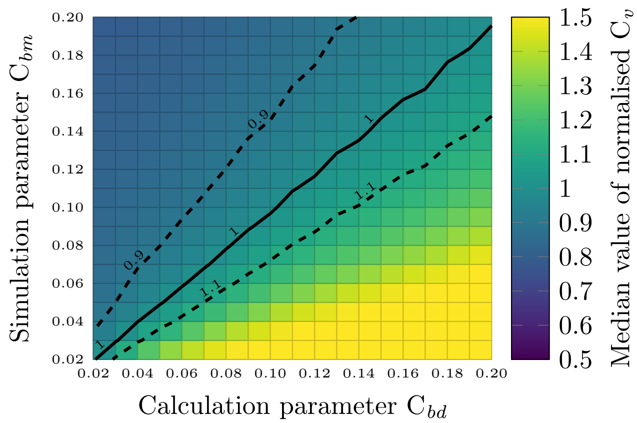


(c) C_p/C_v=1.5

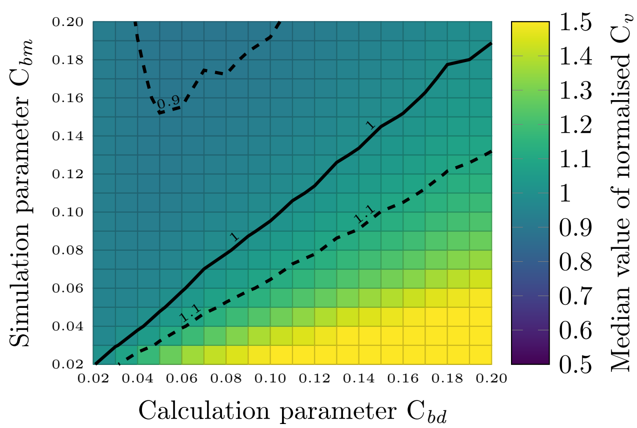
Fig. 15. Effect of the step size (with the C_p/C_v ratio) on the median value mappings of C_v distributions determined using 1000 simulated staircases of $n = 100$ specimens - Sensitivity of the C_{bd} parameter on the fatigue strength calculation method using an initial distribution with C_v=0.05



(a) C_p/C_v=0.5

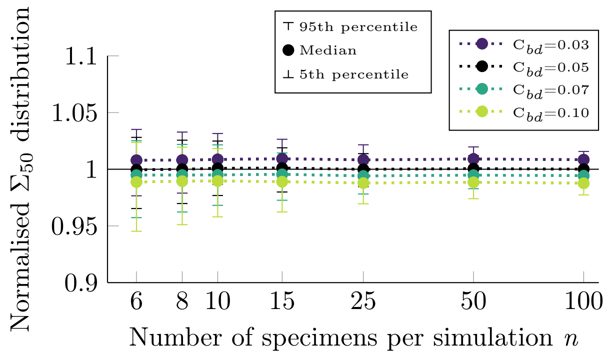


(b) C_p/C_v=1.0

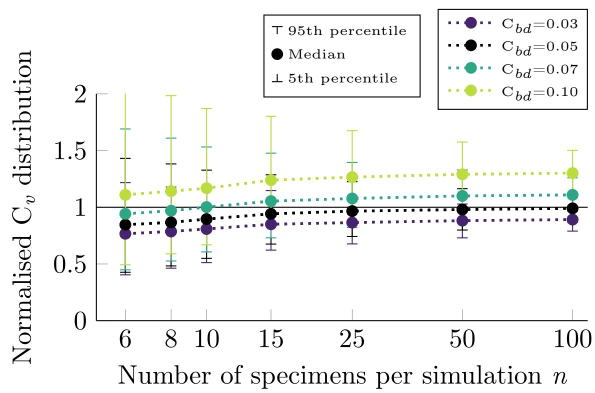


(c) C_p/C_v=1.5

Fig. 16. Effect of the step size (with the C_p/C_v ratio) on the median value mappings of C_v distributions determined using 1000 simulated staircases of $n = 100$ specimens - Sensitivity of the C_{bd} parameter on the fatigue strength calculation method using an initial distribution with C_v=0.1



(a) Resulting distributions of the fatigue limit Σ_{50}

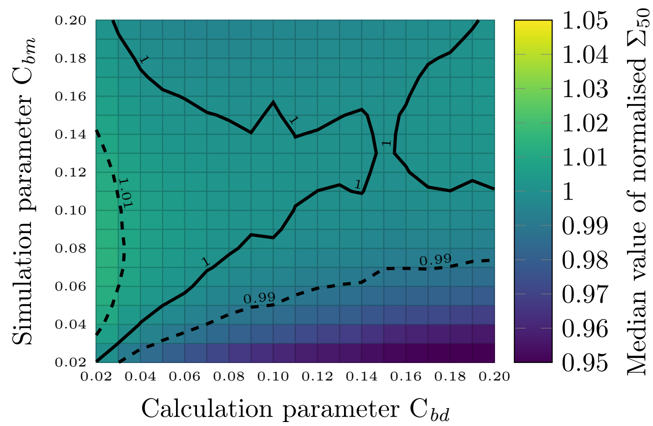


(b) Resulting distributions of the coefficient of variation C_v

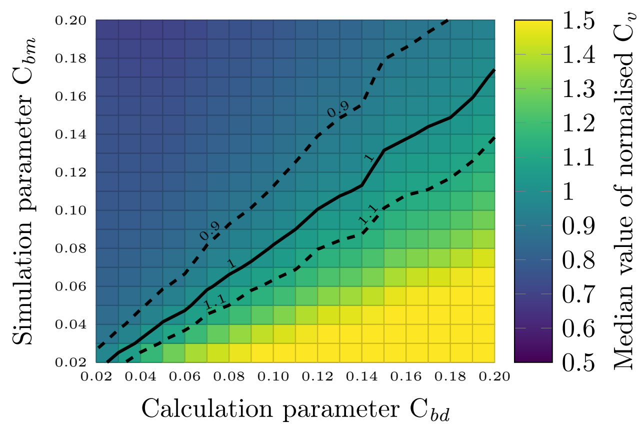
Fig. 17. Distributions of the fatigue parameters determined using 1000 simulated staircases, depending on the number of specimens per simulation - Effect of an imperfect fatigue strength calculation, with different values of C_{bd} (using C_v=0.05, C_p/C_v=1 and C_{bm}=0.05)

and the sensitivity are really similar for the two sample sizes. In fact, a slight difference is only observed for the mappings of the C_v determination (Fig. 18b), the case where the normalised value is 1 is slightly shifted. It means that the best calculation parameter with small sample sizes is no longer for C_{bm}=C_{bd}, but for a lower C_{bd} coefficient (also observed in Fig. 17b). This is explained by two different effects. For a small sample size, the determined fatigue scatter tends to be underestimated (Fig. 4b). Indeed, during the random selection of the fatigue strengths, the distribution tails are not correctly represented. This phenomenon is compensated by the fact that when the C_{bd} parameter is lower than the C_{bm} parameter, the fatigue scatter tends to be overestimated.

Moreover, with this method, the fatigue scatter calculation is nearly always possible, and the uncertainties are similar to the ideal case. However, the median response of the determination method is affected if the slope of the S-N curve is not correctly determined, even with a large number of specimens. A higher



(a) Mapping of the median values of Σ_{50} distributions



(b) Mapping of the median values of C_v distributions

Fig. 18. Effect of the number of specimens on the median value mappings of the distribution of the two fatigue parameters determined using 1000 simulated staircases of $n = 15$ specimens using C_v=0.05 and C_p/C_v=1

C_{bd} coefficient than a C_{bm} coefficient tends to underestimate the median fatigue limit and overestimate the fatigue scatter.

7. Conclusions

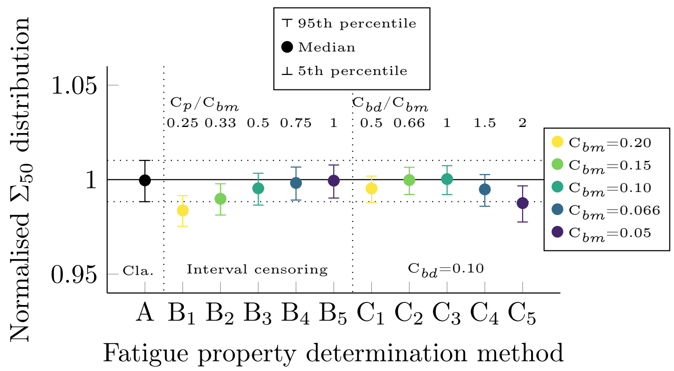
This simulation-based analysis has led to the evaluation of a new staircase procedure with the reuse of unbroken specimens. Firstly, the possible gain has been demonstrated and two different evaluation techniques have been defined: using censoring data or fatigue strength estimation.

These fatigue determination methods have been evaluated using damage consideration. Numerous staircase simulations have been performed in order to estimate the effect of the cumulative damage on the median response and on the scatter in results of the evaluation technique. For the interval censoring method, a domain of validity is defined. It represents the set of parameters for which the cumulative damage does not affect the staircase proceedings and so the censored observations are nearly the same as for the case without damage consideration. For the fatigue strength calculation method, the sensitivity of the parameter representing the slope of the S-N curve is studied.

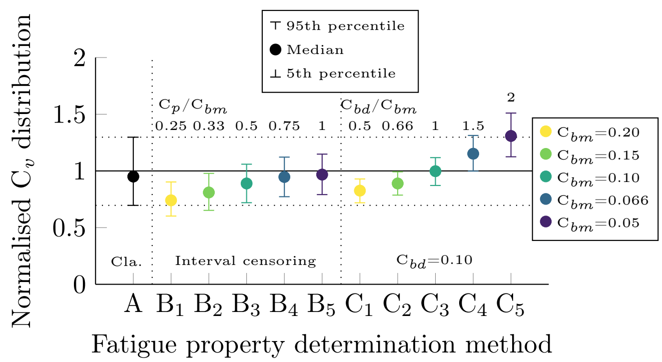
In conclusion, the uncertainties on the fatigue property parameters have been quantified for a large number of configurations, using normalised and linear coefficients. The results are idealised, due to the simulation-based investigation and its assumptions (no noise, distribution shape known, perfect fatigue tests). The highlights of these studies are listed:

- An ideal case has been studied. It represents the case for which the maximum amount of information is extracted from the sample of n specimens, and so, it corresponds to the lowest achievable uncertainties for this sample size. The classic procedure is also studied to get reference results.
- The new staircase procedure, with the reuse of unbroken specimens, brings more information from the sample, in comparison with the classic staircase procedure. However, the damage accumulation must be taken into consideration. In order to study its impact, staircases with damage accumulation have been simulated.
- The censored-data method (and more specifically the use of the interval censoring) is really interesting because almost no assumption is needed and it is a very simple analysis method. Of course, the step size of the staircase procedure must be chosen in the validity domain for which the cumulative damage does not affect the staircase proceedings.
- If the damage accumulation can be estimated, it is possible to make more assumptions for the fatigue analysis in order to further reduce the uncertainties. Indeed, with a fatigue strength calculation method, the maximum amount of data is extracted from the sample, and the results are thus close to the ideal case (if the assumptions are corrects).

In order to compare the two determination methods and evaluate the gain from the classic procedure, the distributions of the parameters estimated from 1000 samples of 100 specimens are plotted (Fig. 19). The same initial distribution is used with C_v=0.05 and with the same step size for the staircase procedure C_p/C_v=1.0. Five conditions of damage accumulation are defined with different C_{bm} and are compared to the classic procedure. There is no damage accumulation in the classic procedure because no reloading is performed (method A), so the results are the same for the five conditions. For the interval censoring method (method B), no damage accumulation is considered during the analysis. The five conditions of damage accumulation are chosen to describe a large range of C_p/C_{bm}. For the fatigue strength determination method, the coefficient C_{bd} is set (method C), and the five conditions describe a large range of C_{bd}/C_{bm}.



(a) Resulting distributions of the fatigue limit Σ_{50}



(b) Resulting distributions of the coefficient of variation C_v

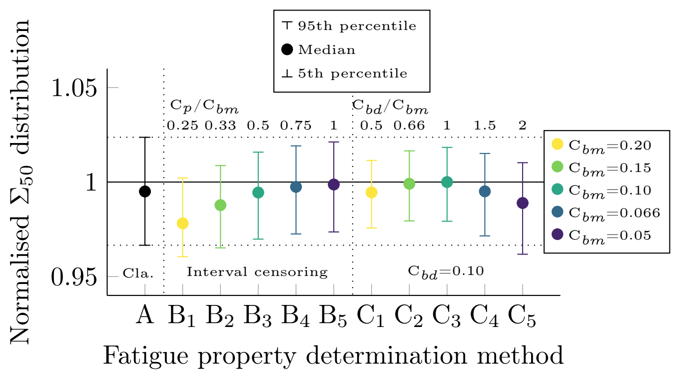
Fig. 19. Distributions of the fatigue parameters determined using 1000 simulated staircases with $n = 100$, depending on the determination method - The classic procedure is compared to the interval censoring method and the fatigue strength calculation method with a set parameter C_{bd}=0.1 (using C_v=0.05, C_p/C_v=1.0 and different values of the simulation parameters C_{bm})

The scatters in the results are significantly reduced when reloading specimens. Indeed, more data are extracted from the sample with the same number of specimens. However, when performing reloading on unbroken specimens, the damage accumulation must be analysed and taken into account. If the condition $C_p/C_{bm} \geq 0.5$ is not fulfilled, the median responses of the interval censoring method are lower than the expected values, even for a large number of specimens. Concerning the fatigue strength calculation method (method C), the median responses are also strongly affected by an incorrect estimation of the slope of the S-N curve. The ideal case is represented by C_{bd}/C_{bm}=1 (method C_3), it represents the lowest possible quantity of uncertainties with this number of specimens.

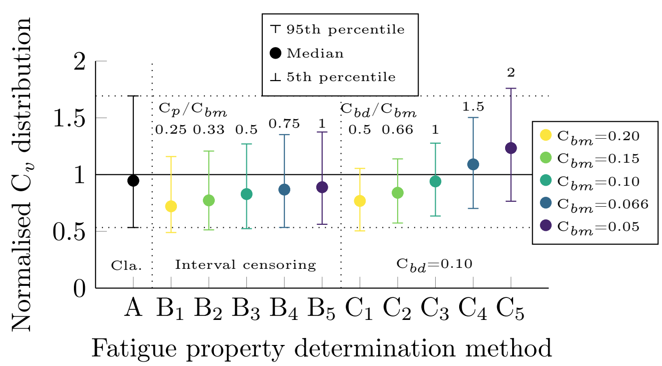
However, during real staircase tests, a lower number of specimens is usually used. The same comparison is presented using 15 specimens per simulated staircase (Fig. 20). The resulting distributions for the coefficient of variation must be analysed with the proportion of possible C_v calculation, shown with the different characteristic values of the distributions (Table 5).

The same conclusions are formulated. The scatters in results are significantly reduced when reloading specimens, but the median responses can be affected by the different determination methods. Moreover, in the domain of validity of the interval censoring method, the C_v calculation is nearly always possible and the scatter in results for the determination of the dispersion is reduced. The fatigue strength calculation method is really interesting in further reducing the scatter in results. However, more assumptions are required for the S-N curve shape and the damage accumulation rule.

The uncertainties for the fatigue scatter are still really numerous for a small number of specimens, even in this simulation-based investigation with idealised



(a) Resulting distributions of the fatigue limit Σ_{50}



(b) Resulting distributions of the coefficient of variation C_v

Fig. 20. Distributions of the fatigue parameters determined using 1000 simulated staircases with $n = 15$, depending on the determination method - The classic procedure is compared to the interval censoring method and the fatigue strength calculation method with a set parameter C_{bd}=0.1 (using C_v=0.05, C_p/C_v=1.0 and different values of the simulation parameters C_{bm})

Table 5
Comparison of resulting distributions of the fatigue parameters for $n = 15$, depending on the determination method using C_v=0.05 and C_p/C_v=1.0

Method		C_{bm}	Normalised Σ_{50} Median	(5th;95th) percentiles	Normalised C_v Proportion (1000 sim.)	Median	(5th;95th) percentiles
Classic procedure	**A**		0.995	(0.967;1.024)	0.73	0.95	(0.53;1.69)
Interval censoring	$\mathbf{B_1}$	0.20	0.978	(0.960;1.002)	0.76	0.72	(0.49;1.16)
	$\mathbf{B_2}$	0.15	0.988	(0.965;1.009)	0.90	0.77	(0.51;1.21)
	$\mathbf{B_3}$	0.10	0.995	(0.970;1.017)	0.96	0.83	(0.52;1.27)
	$\mathbf{B_4}$	0.066	0.997	(0.973;1.019)	0.98	0.87	(0.54;1.35)
	$\mathbf{B_5}$	0.05	0.999	(0.974;1.021)	0.98	0.89	(0.56;1.37)
Adjusted with C_{bs}=0.10	$\mathbf{C_1}$	0.20	0.995	(0.976;1.012)	1	0.77	(0.51;1.05)
	$\mathbf{C_2}$	0.15	0.999	(0.979;1.017)	1	0.84	(0.57;1.14)
	$\mathbf{C_3}$	0.10	1.000	(0.979;1.018)	1	0.94	(0.63;1.28)
	$\mathbf{C_4}$	0.066	0.995	(0.971;1.015)	1	1.09	(0.70;1.50)
	$\mathbf{C_5}$	0.05	0.989	(0.962;1.010)	1	1.23	(0.76;1.76)

results. In an industrial context, the uncertainties for these parameters should be estimated for the sake of part reliability. A simulation-based investigation is a good way to estimate the uncertainties linked to a specific fatigue test procedure and analysis method. Moreover, other parameters can be implemented in the likelihood estimation function, in order to estimate other aspects such as the S-N curve shape. It is already the basis of some evaluation techniques, to directly evaluate the uncertainties on the S-N curve determination [18, 19].

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